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Heterotic vortex strings

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Abstract: We determine the low-energy $\mathcal{N} = (0, 2)$ worldsheet dynamics of vortex strings in a large class of non-Abelian $\mathcal{N} = 1$ supersymmetric gauge theories.

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Contents

1. Introduction

Vortex strings provide a map between four-dimensional non-Abelian gauge theories and two-dimensional sigma-models. The four-dimensional theories in question have a $\mathrm{U}(N_c)$ gauge group and a sufficient number of scalar fields to allow complete gauge symmetry breaking, so that the system lies in the Higgs phase. Theories with this property admit vortex strings. The embedding of the vortex within the non-Abelian gauge group endows the string with a number of orientation modes which parameterize the complex projective space $\mathbf{CP}^{N_c-1}$. Further bosonic and fermionic zero modes of the vortex live in line bundles over $\mathbf{CP}^{N_c-1}$. In this manner, the low-energy dynamics of a single, straight, infinite vortex string is described by some variant of the $\mathbf{CP}^{N_c-1}$ sigma-model living on the $d = 1+1$ dimensional worldsheet [1, 2].

When the four-dimensional gauge theory has $\mathcal{N} = 2$ supersymmetry, a pleasing story emerges. The strings are 1/2-BPS, ensuring that the worldsheet dynamics inherits $\mathcal{N} = (2,2)$ supersymmetry. It was shown in [3, 4], following earlier work of [5, 6], that the quantum dynamics of the worldsheet theory encodes quantitative information about the quantum dynamics of the parent four-dimensional theory, including the Seiberg-Witten curve and the exact BPS mass spectrum. More recently, the correspondence was extended to superconformal points, with a matching between the scaling dimensions of chiral primary operators in the four-dimensional bulk and on the worldsheet [7]. For a review of the classical and quantum dynamics of these strings, see [8].

The purpose of this paper is to present a detailed study of the classical dynamics of vortex strings in $\mathcal{N} = 1$ four-dimensional gauge theories. For certain choices of parameters the strings once again preserve 1/2 of supersymmetry, now guaranteeing $\mathcal{N} = (0,2)$ supersymmetry on the worldsheet. For this reason, we refer to vortices in $\mathcal{N} = 1$ theories as "heterotic vortex strings". We will determine the explicit $\mathcal{N} = (0,2)$ $\mathbf{CP}^{N_c-1}$ sigma-models, and their variations, which describe the low-energy dynamics of vortex strings in a large class of $\mathcal{N} = 1$ gauge theories.[1]

The paper is organized as follows: section 2 contains a detailed discussion of the $\mathcal{N} = (2,2)$ worldsheet dynamics of vortex strings in $\mathcal{N} = 2$ four-dimensional theories. This section is mostly a review of previous work, although explicit expressions for bosonic and fermionic zero modes are provided which generalize results in the literature from $\mathrm{U}(2)$ gauge theories to $\mathrm{U}(N_c)$ gauge theories. Particular attention is paid to the chirality of different fermionic zero modes since this will prove important in later sections. Section 3 also contains review material, describing the basics of the superfield formalism for $\mathcal{N} = (0,2)$ supersymmetry in $d = 1+1$ dimensions.

The meat of the paper is in section 4. We consider two different classes of deformations, each of which breaks the four-dimensional supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ through the introduction of a superpotential for the adjoint chiral multiplet. In each case, we show that there is a unique $\mathcal{N} = (0,2)$ worldsheet theory which correctly captures all BPS

[1] Vortex strings in various non-Abelian theories with less supersymmetry were previously studied in [9-13] and in some cases qualitative agreement was found between the dynamics of the worldsheet theory and the bulk. We will comment more on the relationship of our work to some of these papers in section 4.

properties of the vortex string and predicts the interaction of fermionic zero modes. We also include an appendix which collates the notation for bulk and worldsheet fields used throughout the paper.

2. The $\mathcal{N} = (2,2)$ dynamics of vortex strings

In this section we review the dynamics of vortex strings in four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry. The vortices are 1/2-BPS, ensuring that the $d = 1+1$ dimensional worldsheet dynamics of the string inherits $\mathcal{N} = (2,2)$ supersymmetry.

2.1 The four-dimensional theory

Our starting point is the $d = 3+1$, $\mathcal{N} = 2$ supersymmetric $\mathrm{U}(N_c)$ gauge theory, with N_f flavors transforming in the fundamental representation.[2] We describe the theory in the language of four-dimensional $\mathcal{N} = 1$ superfields. The $\mathcal{N} = 2$ vector multiplet consists of an $\mathcal{N} = 1$ vector multiplet V and an $\mathcal{N} = 1$ adjoint chiral multiplet A. Similarly, each flavor hypermultiplet splits into two chiral multiplets, Q_i and $\tilde{Q}_i$ where $i = 1, \ldots, N_f$ is the flavor index. Each Q_i transforms in the fundamental $\mathbf{N}_c$ of the gauge group, while each $\tilde{Q}_i$ transforms in the anti-fundamental $\bar{\mathbf{N}}_c$. We denote the complexified gauge coupling of the theory as

$$\tau = \frac{2\pi i}{e^2} + \frac{\theta}{2\pi} \ . \tag{2.1}$$

The four dimensional theory has the usual superpotential required for $\mathcal{N} = 2$ supersymmetry,

$$\mathcal{W}_{\mathcal{N}=2} = \sqrt{2} \sum_{i=1}^{N_f} \tilde{Q}_i A Q_i \ . \tag{2.2}$$

The scalar potential of the theory is dictated by the D-term and the F-terms arising from this superpotential. In components it is given by,

$$\begin{aligned} V_{4d} = & \frac{e^2}{2} \mathrm{Tr} \left(\sum_{i=1}^{N_f} Q_i Q_i^\dagger - \tilde{Q}_i \tilde{Q}_i^\dagger - v^2 \, 1_{N_c} \right)^2 + e^2 \mathrm{Tr} | \sum_{i=1}^{N_f} \tilde{Q}_i Q_i |^2 \\ & + \sum_{i=1}^{N_f} \left(Q_i^\dagger \{A, A^\dagger\} Q_i + \tilde{Q}_i \{A, A^\dagger\} \tilde{Q}_i^\dagger \right) + \frac{1}{2e^2} \mathrm{Tr} |[A, A^\dagger]|^2 \end{aligned} \tag{2.3}$$

where we have taken the liberty of denoting the component scalar fields by the same Roman letter as the superfield in which they reside. We have included a D-term Fayet-Iliopoulos (FI) parameter v^2 for the central $\mathrm{U}(1) \subset \mathrm{U}(N_c)$. This is consistent with $\mathcal{N} = 2$

[2] Conventions: We pick Hermitian generators T^m with Killing form $\mathrm{Tr}\, T^m T^n = \frac{1}{2}\delta^{mn}$. We write the gauge field as $A_\mu = A_\mu^m T^m$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. Fundamental covariant derivatives are $\mathcal{D}_\mu Q = \partial_\mu Q - i A_\mu Q$; adjoint covariant derivatives are $\mathcal{D}_\mu A = \partial_\mu A - i[A_\mu, A]$. Our summation conventions are inconsistent: a sum over repeated indices is usually left implicit unless there is some ambiguity or a point that requires emphasis.

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supersymmetry and forces the theory into the Higgs phase, with Q_i gaining a vacuum expectation value (vev). For $N_f < N_c$, the rank condition ensures the D-term cannot vanish and there is no supersymmetric ground state. We do not consider this case. When $N_f > N_c$, the D-term and F-term conditions do not fix the vevs of Q_i and $\tilde{Q}_i$ completely and there is a Higgs branch of vacua; we shall discuss this situation in section 2.5.2. For now we restrict attention to the case $N_f = N_c$ for which there is a unique supersymmetric ground state in which the gauge group is completely broken. Up to a gauge transformation the ground state is given by

$$Q^a{}_i = v\delta^a{}_i \quad , \quad \tilde{Q}_i = A = 0 \tag{2.4}$$

where $a = 1, \ldots, N_c$ is the color index. The theory lies in the color-flavor locked phase, with the vacuum expectation value preserved by a simultaneous gauge and flavor rotation. The symmetry breaking pattern is thus broken to the diagonal combination of the two (recall that we are looking at the theory with $N_f = N_c$)

$$\mathrm{U}(N_c) \times \mathrm{SU}(N_f) \to \mathrm{SU}(N_c)_{\mathrm{diag}} \ . \tag{2.5}$$

2.2 The vortex

The central $\mathrm{U}(1) \subset \mathrm{U}(N_c)$ does not survive the symmetry breaking (2.5), a fact which provides sufficient topology to ensure the presence of vortex strings in the theory [14]. These vortices preserve 1/2 of the supersymmetry, ensuring $\mathcal{N} = (2,2)$ supersymmetric dynamics on their $d = 1+1$ dimensional worldvolume. Infinite, straight strings oriented in the x^3 direction satisfy the first-order equations,

$$\begin{aligned} F_{12} &= e^2\left(\sum_{i=1}^{N_f} Q_i Q_i^\dagger - v^2\, 1_{N_c}\right) \\ \mathcal{D}_z Q_i &\equiv \frac{1}{2}(\mathcal{D}_1 Q_i - i\mathcal{D}_2 Q_i) = 0 \end{aligned} \tag{2.6}$$

where $z = x^1 + ix^2$ parameterizes the transverse plane. These are the non-Abelian vortex equations. Solutions to these equations have tension

$$T_k = 2\pi k v^2 \tag{2.7}$$

where $k = -\mathrm{Tr} \int (F_{12}/2\pi) \in \mathbf{Z}^+$ is the winding number.

Solutions to the vortex equations with winding number k have $2kN$ bosonic collective coordinates. For a single $k = 1$ vortex, they break down as follows: there are 2 collective coordinates corresponding to the position of the string in the $z = x^1 + ix^2$ plane. The remaining $2(N-1)$ collective coordinates are Goldstone modes arising from the action of the surviving symmetry (2.5) on the vortex string. They parameterize $\mathrm{SU}(N_c)/[\mathrm{SU}(N_c - 1) \times \mathrm{U}(1)] \cong \mathbf{CP}^{N_c-1}$ [1, 2].

An explicit realization of the orientational modes is most simply given in *singular gauge* in which Q does not wind asymptotically, with the flux instead arising from a singular gauge

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potential [2]. Suppose that the Abelian $N_c = 1$ vortex equations are solved by two profile functions $q(\rho)$ and $a(\rho)$, where $\rho = \sqrt{(x^1)^2 + (x^2)^2}$ is the radial distance from the string

$$Q_{\text{Abelian}} = vq(\rho) \quad \text{and} \quad (A_z)_{\text{Abelian}} = -i\bar{z}a(\rho) \ . \tag{2.8}$$

Here the complexified gauge connection is $A_z = \frac{1}{2}(A_1 - iA_2)$. Plugging this ansatz into the vortex equations gives two first order ordinary differential equations,

$$q' = 2\rho a q \quad \text{and} \quad 4a + 2\rho a' = e^2 v^2 (q^2 - 1) \tag{2.9}$$

with prime denoting the derivative with respect to ρ. These equations are known to admit a unique solution satisfying the appropriate boundary conditions,

$$q(\rho) \to \begin{cases} 1 \\ 0 \end{cases} , \quad a(\rho) \to \begin{cases} 0 & \text{as } \rho \to \infty \\ 1/2\rho^2 & \text{as } \rho \to 0 \ . \end{cases} \tag{2.10}$$

However, the solution does not have a simple analytic form. A sketch of the profile $q(\rho)$ is shown in figure 1.

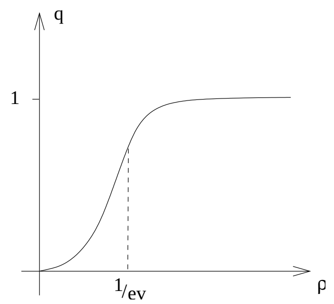

Figure 1: The vortex profile.

With the $k = 1$ Abelian vortex solution in hand, one may simply construct a solution to the non-Abelian equations by embedding thus,

$$Q^a{}_i = \left(\frac{\phi^a \bar{\phi}_i}{r}\right) v[q(\rho) - 1] + v\delta^a{}_i \quad \text{and}$$
$$(A_z)^a{}_b = -i\bar{z}a(\rho) \left(\frac{\phi^a \bar{\phi}_b}{r}\right) . \tag{2.11}$$

The $\phi^a \in \mathbf{C}^{N_c}$ define the orientation of the vortex in the gauge and flavor groups. In order that this reduce to the Abelian solution, we require

$$\sum_{a=1}^{N_c} |\phi^a|^2 = r \tag{2.12}$$

with r a constant that will be fixed shortly. The solutions (2.11) are invariant under the simultaneous rotation,

$$\phi^a \to e^{i\alpha} \phi^a \ . \tag{2.13}$$

The ϕ^a, subject to the constraint (2.12) and identification (2.13), provide homogeneous coordinates on $\mathbf{CP}^{N_c-1}$. The $\mathrm{SU}(N_c)$ symmetry of four-dimensions descends to the vortex string, with the ϕ^a transforming in the fundamental representation. This ensures that the $\mathbf{CP}^{N_c-1}$ is endowed with the symmetric Fubini-Study metric. The Kähler class of this space is r.

A comment on notation: since $N_f = N_c$, both $Q^a{}_i$ and $(A_z)^a{}_b$ are $N_c \times N_c$ matrices. In what follows, we shall often neglect to write the indices on both. In this notation, Q is a matrix on which gauge rotations $U \in \mathrm{U}(N_c)$ act from the left, while flavor rotations $V \in \mathrm{SU}(N_f)$ act from the right, so that $Q \to UQV^\dagger$.

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2.2.1 Bosonic zero modes

For general winding number k, the vortex zero modes are defined to be solutions to the linearized vortex equations,

$$\begin{aligned}\mathcal{D}_z\delta A_{\bar{z}} - \mathcal{D}_{\bar{z}}\delta A_z &= \frac{ie^2}{2}(\delta Q Q^\dagger + Q\delta Q^\dagger)\\ \mathcal{D}_z\delta Q &= i\delta A_z Q\ .\end{aligned} \tag{2.14}$$

These are to be supplemented with a suitable gauge fixing condition which is derived from Gauss' law and reads

$$\mathcal{D}_z\delta A_{\bar{z}} + \mathcal{D}_{\bar{z}}\delta A_z = -\frac{ie^2}{2}(\delta Q Q^\dagger - Q\delta Q^\dagger)\ . \tag{2.15}$$

This gauge fixing condition combines with the first of the linearized vortex equations to leave us with two, complex, first order equations to be solved around the background of a fixed vortex configuration,

$$\begin{aligned}2\mathcal{D}_{\bar{z}}\delta A_z &= -ie^2\delta Q Q^\dagger\\ \mathcal{D}_z\delta Q &= i\delta A_z Q.\end{aligned} \tag{2.16}$$

We now derive the solutions to these equations that arise from the symmetries of the system.

Translational mode. For any winding number k, the two translational modes are always given by

$$\delta A_z = F_{\bar{z}z} \quad \text{and} \quad \delta Q = \mathcal{D}_{\bar{z}} Q \tag{2.17}$$

which can be checked to satisfy (2.16) using the fact that the background fields obey the second order equations of motion.

Orientational modes. The zero modes corresponding to orientation are only slightly more complicated. In general they can be written as

$$\begin{aligned}\delta A_z &= \mathcal{D}_z\Omega\\ \delta Q &= i(\Omega Q - Q\hat{\Omega}).\end{aligned} \tag{2.18}$$

Here $\Omega(x)$ is an infinitesimal gauge rotation, while $\hat{\Omega}$ is an infinitesimal flavor rotation. Since only the diagonal subgroup (2.5) of these is preserved in the vacuum, we require that $\Omega(x) \to \hat{\Omega}$ as $x \to \infty$. In terms of our orientation coordinates ϕ^i, this diagonal rotation can be written as,

$$\hat{\Omega}^i{}_j = -i\left[\delta\phi^i\bar{\phi}_j - \phi^i\delta\bar{\phi}_j - 2iu\phi^i\bar{\phi}_j\right] \tag{2.19}$$

which holds for any u. Requiring that $\hat{\Omega} \in su(N_c)$ fixes u to be

$$u = -i\bar{\phi}_i\,\delta\phi^i\ . \tag{2.20}$$

Later u will become a gauge field on the worldsheet whose role is to implement the identification (2.13). For now, we can treat u as a connection and introduce the covariant variation $\nabla\phi^i = \delta\phi^i - iu\phi^i$ which satisfies $\nabla\phi^i \cdot \bar{\phi}_i = 0$. In this notation

$$\hat{\Omega}^i{}_j = -i[(\nabla\phi^i)\bar{\phi}_j - \phi^i\nabla\bar{\phi}_j]. \tag{2.21}$$

The zero mode equations (2.16) translate to the requirement that $\Omega(x)$ satisfy the second order differential equation

$$\mathcal{D}^2\Omega = e^2\left[\{\Omega, QQ^\dagger\} - 2Q\hat{\Omega}Q^\dagger\right]. \tag{2.22}$$

Everything above holds for arbitrary winding number k. For a single vortex, with $k=1$, the solution to (2.22) was provided in [10] (see equation (28) of that paper) and depends only on the profile function $q(\rho)$ of the vortex[3]

$$\Omega(\rho) = q(\rho)\,\hat{\Omega}. \tag{2.23}$$

Using the solution (2.23), we can now be more explicit about the orientation zero modes for a single vortex. Making use of the vortex equations (2.6), we find

$$\begin{aligned}(\delta A_z)^a{}_b &= -2i(\partial_z q)\,(\nabla\phi^a)\bar{\phi}_b \\ \delta Q^a{}_i &= v(q^2-1)\,(\nabla\phi^a)\bar{\phi}_i.\end{aligned} \tag{2.24}$$

2.3 Fermions

We now turn to a study of the fermionic zero modes [15]. We start by describing the Dirac equations in four-dimensions and their solutions for a single $k=1$ vortex. We will pay particular attention to the correlation between the chirality of the worldsheet and four-dimensional fermions.

In the following we use four-dimensional Weyl fermions ψ_α and $\bar{\lambda}^{\dot{\alpha}}$ with $\alpha, \dot{\alpha} = 1,2$. The notation is standard Wess and Bagger fare [16] with, for example, $\psi\lambda = \psi^\alpha\lambda_\alpha = \lambda\psi$ and $\bar{\psi}\bar{\lambda} = \bar{\psi}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} = \bar{\lambda}\bar{\psi}$. Indices are raised and lowered with $\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = i\sigma_2$. Our signature is mostly minus and we define $(\sigma^\mu)_{\alpha\dot{\alpha}} = (-1, \sigma^i)$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = (-1, -\sigma^i)$.

The $\mathcal{N}=2$ vector multiplet in four dimensions contains two Weyl fermions, λ and η, each transforming in the adjoint representation of the $\mathrm{U}(N_c)$ gauge group. The fermion λ lives in the $\mathcal{N}=1$ vector multiplet while η lives in the adjoint chiral multiplet A. Each hypermultiplet also contains two Weyl fermions, ψ and $\tilde{\psi}$. These live in Q and $\tilde{Q}$, and transform in the $\mathbf{N}_c$ and $\bar{\mathbf{N}}_c$ representations respectively. The Dirac equations in the $\mathcal{N}=2$ theory are

$$\begin{aligned}
-\frac{i}{e^2}\,\bar{\not{D}}\lambda - \frac{i\sqrt{2}}{e^2}[\bar{\eta}, A] + i\sqrt{2}Q_i\bar{\psi}_i - i\sqrt{2}\bar{\tilde{\psi}}_i\tilde{Q}_i &= 0 \\
-\frac{i}{e^2}\,\bar{\not{D}}\eta - \frac{i\sqrt{2}}{e^2}[A, \bar{\lambda}] - \sqrt{2}\tilde{Q}_i^\dagger\bar{\psi}_i - \sqrt{2}\bar{\tilde{\psi}}_i Q_i^\dagger &= 0 \\
-i\,\bar{\not{D}}\psi_i + i\sqrt{2}\bar{\lambda}Q_i - \sqrt{2}A^\dagger\bar{\tilde{\psi}}_i - \sqrt{2}\bar{\eta}\tilde{Q}_i^\dagger &= 0 \\
-i\,\bar{\not{D}}\tilde{\psi}_i - i\sqrt{2}\tilde{Q}_i\bar{\lambda} - \sqrt{2}\bar{\psi}_i A^\dagger - \sqrt{2}Q_i^\dagger\bar{\eta} &= 0.
\end{aligned} \tag{2.25}$$

[3]Equation (2.23) solves (2.22) by virtue of the vortex profile obeying the second order equation $4\partial_z\partial_{\bar{z}}q - 4a^2\rho^2 q = e^2v^2q(q^2-1)$.

We wish to study these equations in the background of the vortex. Here they simplify considerably since we have $A = \tilde{Q}_i = 0$. The equations decouple into two pairs: the first set of equations are for λ and $\bar{\psi}_i$

$$-\frac{i}{e^2}\,\bar{\not{\mathcal{D}}}\lambda + i\sqrt{2}Q_i\bar{\psi}_i = 0 \quad \text{and} \quad -i\,\not{\mathcal{D}}\bar{\psi}_i - i\sqrt{2}Q_i^\dagger\lambda = 0. \tag{2.26}$$

The second set of equations are for η and $\bar{\tilde{\psi}}_i$,

$$-\frac{i}{e^2}\,\bar{\not{\mathcal{D}}}\eta - \sqrt{2}\bar{\tilde{\psi}}_iQ_i^\dagger = 0 \quad \text{and} \quad -i\,\not{\mathcal{D}}\bar{\tilde{\psi}}_i - \sqrt{2}\eta Q_i = 0. \tag{2.27}$$

2.3.1 Chirality

Each pair of four-dimensional fermions gives rise to a fermi zero mode on the vortex string of a specific chirality. Since this will be important in later sections, we dwell on the point a little here. The first step is to see which components of the spinors can turn on in the background of a vortex or anti-vortex. We will need the following identities,

$$\not{\mathcal{D}}_{\alpha\dot{\alpha}} \equiv (\sigma^\mu)_{\alpha\dot{\alpha}}\mathcal{D}_\mu = 2\begin{pmatrix} -\mathcal{D}_- & \mathcal{D}_z \\ \mathcal{D}_{\bar{z}} & -\mathcal{D}_+ \end{pmatrix} \quad \text{and} \quad \bar{\not{\mathcal{D}}}^{\dot{\alpha}\alpha} \equiv (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\mathcal{D}_\mu = -2\begin{pmatrix} \mathcal{D}_+ & \mathcal{D}_z \\ \mathcal{D}_{\bar{z}} & \mathcal{D}_- \end{pmatrix} \tag{2.28}$$

where $\mathcal{D}_\pm = \frac{1}{2}(\mathcal{D}_0 \pm \mathcal{D}_3)$ and $\mathcal{D}_z = \frac{1}{2}(\mathcal{D}_1 - i\mathcal{D}_2)$ and $\mathcal{D}_{\bar{z}} = \frac{1}{2}(\mathcal{D}_1 + i\mathcal{D}_2)$. Our strings are static and oriented in the x^3 direction, so in searching for zero modes of the Dirac equation in the presence of a vortex we may initially set $\mathcal{D}_\pm = 0$. We decompose the spinors as $(\lambda_1, \lambda_2) = (\lambda_-, \lambda_+)$ and $(\lambda^1, \lambda^2) = (\lambda^-, \lambda^+)$ so that, with our raising and lowering conventions, $\lambda^+ = -\lambda_-$ and $\lambda^- = \lambda_+$. To see which components turn on in the background of the vortex, we act on the first and second equations in (2.26) with $\not{\mathcal{D}}$ and $\bar{\not{\mathcal{D}}}$ respectively. Making use of the vortex equation $\mathcal{D}_zQ_i = 0$, we find

$$\begin{aligned} \Big(-\frac{4}{e^2}\mathcal{D}_z\mathcal{D}_{\bar{z}} + 2Q_iQ_i^\dagger\Big)\lambda_- &= 0 \\ \Big(-\frac{4}{e^2}\mathcal{D}_{\bar{z}}\mathcal{D}_z + 2Q_iQ_i^\dagger\Big)\lambda_+ - \sqrt{2}(\mathcal{D}_{\bar{z}}Q_i)\,\bar{\psi}_{+i} &= 0 \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} (-\mathcal{D}_z\mathcal{D}_{\bar{z}}\delta_{ij} + 2Q_iQ_j^\dagger)\bar{\psi}_{+j} - \sqrt{2}(\mathcal{D}_zQ_i^\dagger)\,\lambda_+ &= 0 \\ (-\mathcal{D}_{\bar{z}}\mathcal{D}_z\delta_{ij} + 2Q_i^\dagger Q_j)\bar{\psi}_{-j} &= 0. \end{aligned} \tag{2.30}$$

The operators appearing in the equations for λ_- and $\bar{\psi}_{-i}$ are positive definite: these components can have no zero modes. All zero modes live in the components λ_+ and $\bar{\psi}_{+i}$.

To see how this correlates with the chirality of the worldsheet fermions, we now allow these zero modes to vary along the string so that $\lambda_+ = \lambda_+(x^0, x^3)$ and $\bar{\psi}_{+i} = \bar{\psi}_{+i}(x^0, x^3)$. Plugging this ansatz back into the Dirac equation, including now the derivatives $\mathcal{D}_\pm$ in (2.28), we find the equations of motion $\partial_-\lambda_+ = \partial_-\bar{\psi}_{+i} = 0$. We call these fermions *right movers*.

Repeating this analysis for Dirac equations (2.27), we find that η_- and $\bar{\tilde{\psi}}_{-i}$ both carry zero modes in the background of the vortex. They are *left movers* on the string worldsheet[4]

[4] In the background of an anti-vortex, with $\mathcal{D}_{\bar{z}}Q_i = 0$, the chirality of the fermi zero modes is reversed, so that $(\lambda, \bar{\psi})$ donate left movers, while $(\eta, \bar{\tilde{\psi}})$ donate right movers.

2.3.2 Fermi zero modes

From the previous analysis, we learn that the right moving fermi zero modes solve

$$\begin{aligned}\sqrt{2}\mathcal{D}_z\lambda_+ &= -e^2\, Q_i\bar{\psi}_{+i}\\ \sqrt{2}\mathcal{D}_{\bar{z}}\bar{\psi}_{+i} &= -Q_i^\dagger\lambda_+\end{aligned} \tag{2.31}$$

while the equations for the left moving fermi zero modes solve

$$\begin{aligned}\sqrt{2}i\mathcal{D}_{\bar{z}}\eta_- &= -e^2\bar{\tilde{\psi}}_{-i}Q_i^\dagger\\ \sqrt{2}i\mathcal{D}_z\bar{\tilde{\psi}}_{-i} &= \eta_- Q_i.\end{aligned} \tag{2.32}$$

Each of these pairs of equations is the same as the equations for bosonic zero modes (2.16) that are derived by linearizing the vortex equations and imposing a gauge fixing constraint. The relationship between the bosonic and fermionic zero modes is given by

$$\begin{aligned}\lambda_+ \leftrightarrow \delta A_{\bar{z}} \quad &\text{and} \quad \sqrt{2}\bar{\psi}_{+i} \leftrightarrow -i\delta Q_i^\dagger\\ \eta_- \leftrightarrow \delta A_z \quad &\text{and} \quad \sqrt{2}\bar{\tilde{\psi}}_{-i} \leftrightarrow -\delta Q_i.\end{aligned} \tag{2.33}$$

This mapping between the zero mode profiles is a consequence of the preserved supersymmetry in the background of the vortex. Using this, it is trivial to derive the explicit zero modes in the case of a single $k = 1$ vortex.

Goldstino modes. The bosonic translational modes were given in (2.17). Their fermionic counterparts are

$$\begin{aligned}\lambda_+ = F_{z\bar{z}}\bar{\chi}_+ \quad &\text{and} \quad \bar{\psi}_{+i} = -\frac{i}{\sqrt{2}}\mathcal{D}_z Q_i^\dagger\bar{\chi}_+\\ \eta_- = F_{\bar{z}z}\chi_- \quad &\text{and} \quad \bar{\tilde{\psi}}_{-i} = -\frac{1}{\sqrt{2}}\mathcal{D}_{\bar{z}}Q_i\chi_-.\end{aligned} \tag{2.34}$$

Both of these are Goldstino modes, arising from acting on the bosonic vortex profile (2.11) with the two broken supersymmetries parameterized by $\chi_\pm$. The above formulae hold for arbitrary k; if we restrict to the explicit $k = 1$ solution, we may write these in terms of the vortex profile function $q(\rho)$,

$$\begin{aligned}(\lambda_+)^a{}_b = \frac{ie^2v^2}{2r}(q^2-1)\phi^a\bar{\phi}_i\bar{\chi}_+ \qquad &\text{and} \quad \bar{\psi}^A_{+i} = -\frac{i\sqrt{2}v}{r}(\partial_z q)\,\phi^a\bar{\phi}_b\bar{\chi}_+\\ \eta_- = -\frac{ie^2v^2}{2r}(q^2-1)\phi^a\bar{\phi}_b\chi_- \quad &\text{and} \quad \bar{\tilde{\psi}}^a_{-i} = -\frac{\sqrt{2}v}{r}(\partial_{\bar{z}}q)\,\phi^a\bar{\phi}_i\chi_-.\end{aligned} \tag{2.35}$$

Super-orientation modes. The superpartners of the orientational modes are equally easy to write down. Given the bosonic zero modes (2.18), we have

$$\begin{aligned}(\lambda_+)^a{}_b = 2i(\partial_{\bar{z}}q)\,\phi^a\bar{\xi}_{+b} \quad &\text{and} \quad \bar{\psi}^a_{+i} = -\frac{iv}{\sqrt{2}}(q^2-1)\phi_i\bar{\xi}^a_+\\ (\eta_-)^a{}_b = -2i(\partial_z q)\,\xi^a_-\bar{\phi}_b \quad &\text{and} \quad \bar{\tilde{\psi}}^a_{-i} = -\frac{v}{\sqrt{2}}(q^2-1)\xi^a_-\bar{\phi}_i.\end{aligned} \tag{2.36}$$

It is clear from these expressions, that the redundancy (2.13) which acts among the ϕ_i orientational coordinates, must also act on the superpartners $\xi_{\pm i}$, so that

$$\phi_i \to e^{i\alpha}\phi_i \quad \text{and} \quad \xi_i \to e^{i\alpha}\xi_i. \tag{2.37}$$

Moreover, the fact that there do not exist orientational coordinates in the $N = 1$ Abelian theory means that we must impose a constraint on the $\xi_{\pm i}$, namely

$$\sum_{i=1}^{N_c} \bar{\phi}^i \xi_{\pm i} = 0. \tag{2.38}$$

2.4 Supersymmetric dynamics

The low-energy dynamics of the vortex string arises by promoting the collective coordinates z, $\chi_\pm$, ϕ_i and $\xi_{\pm i}$ to dynamical fields on the string worldsheet, depending on $y^0 \equiv x^0$ and $y^1 \equiv x^3$. The fact that the vortices are BPS, preserving 1/2 of the $\mathcal{N} = 2$ four-dimensional supersymmetry, ensures that the resulting worldsheet dynamics is invariant under $\mathcal{N} = (2,2)$ supersymmetry. Indeed, the various bosonic and fermionic collective coordinates are easily packaged into $\mathcal{N} = (2,2)$ superfields. The translational mode z and the two Goldstino modes $\chi_\pm$ sit in an $\mathcal{N} = (2,2)$ chiral multiplet Z. Our notation is standard[5] and follows, for example, [37]

$$Z = z + \theta^+\chi_+ + \theta^-\chi_- + \theta^+\theta^- G_Z + \dots \ . \tag{2.39}$$

Similarly, the orientation modes ϕ_i and their superpartners $\xi_{\pm i}$ also sit in $(2,2)$ chiral multiplets,

$$\Phi_i = \phi_i + \theta^+\xi_{+i} + \theta^-\xi_{-i} + \theta^+\theta^- G_i + \dots \ . \tag{2.40}$$

The two constraints $\bar{\phi}^i\phi_i = r$, and $\bar{\phi}^i\xi_{\pm i} = 0$, together with the identification (2.37), are imposed on the worldsheet theory by introducing an auxiliary $\mathcal{N} = (2,2)$ vector multiplet which, in Wess-Zumino gauge, has components

$$\begin{aligned} U \ = \ & -\theta^-\bar{\theta}^-(u_0 - u_1) + \theta^+\bar{\theta}^+(u_0 + u_1) - \theta^-\bar{\theta}^+\sigma - \theta^+\bar{\theta}^-\bar{\sigma} \\ & +\sqrt{2}i\theta^i\theta^+(\theta^-\bar{\zeta}_- + \bar{\theta}^+\bar{\zeta}_+) + \sqrt{2}i\bar{\theta}^+\bar{\theta}^-(\theta^-\zeta_- + \theta^+\zeta_+) + 2\theta^-\theta^+\bar{\theta}^+\bar{\theta}^- D. \end{aligned} \tag{2.41}$$

The two dimensional field strength $u_{01} = \partial_0 u_1 - \partial_1 u_0$ is naturally housed in a twisted chiral multiplet, defined by $\Sigma = \bar{D}_+ D_- U/\sqrt{2}$, with component expansion

$$\Sigma = \sigma - i\sqrt{2}\theta^+\bar{\zeta}_+ - i\sqrt{2}\bar{\theta}^-\zeta_- + \sqrt{2}\theta^+\bar{\theta}^-(D - iu_{01}) + \dots \ . \tag{2.42}$$

The fields σ, $\zeta_\pm$ and D are all auxiliary. Their role will become clear shortly.

With the exception of a single integration constant t, the dynamics of a $k = 1$ vortex string is fixed entirely by the symmetries of the theory. In particular, the $\mathrm{SU}(N_c)_{\mathrm{diag}}$

[5] The one deviation from standard notation is to label the complex auxiliary fields in each chiral multiplet as G. This distinguishes them from the auxiliary F fields in four-dimensions.

symmetry of (2.5) descends to an SU(N_c) global symmetry on the worldsheet, under which the Φ_i transform in the fundamental $\mathbf{N_c}$ representation. The resulting dynamics is given by

$$\mathcal{L}_{\rm vortex} = \int d^4\theta\ T\bar{Z}Z + \sum_{i=1}^{N_c} \bar{\Phi}_i e^{2U}\Phi_i + \frac{it}{2\sqrt{2}} \int d\theta^+ d\bar{\theta}^-\ \Sigma \tag{2.43}$$

where $T = 2\pi v^2$ is the tension of the vortex, while

$$t = ir + \frac{\theta}{2\pi} \tag{2.44}$$

is the integration constant that needs to be fixed, and plays the role of a complexified worldsheet FI parameter. After integrating out the auxiliary fields G_Z and G_i, the purely bosonic part of the worldsheet Lagrangian reads,

$$\mathcal{L}_{\rm bose} = T\,|\partial_m z|^2 + \sum_{i=1}^{N_c}\left(|\mathcal{D}_m\phi^i|^2 - 2|\sigma|^2|\phi_i|^2\right) + D\left(\sum_{i=1}^{N_c}|\phi_i|^2 - r\right) + \frac{\theta}{2\pi}u_{01}. \tag{2.45}$$

Here the ϕ_i fields carry charge $+1$ under the gauge symmetry, with $\mathcal{D}\phi_i = \partial\phi_i - iu\phi_i$. Dividing out by this symmetry imposes the identification (2.13): $\phi_i \to e^{i\alpha}\phi_i$. Meanwhile, the D-field in this Lagrangian plays the role of a Lagrange multiplier, imposing the condition (2.12): $\sum_i |\phi_i|^2 = r$. The value of r is fixed by the requirement that the kinetic terms for ϕ_i are canonical. One finds the result,

$$r = \frac{2\pi}{e^2}. \tag{2.46}$$

This result was first shown using a brane construction in [1], and later re-derived by explicitly computing the overlap of zero modes[6] in [10]. Similarly, it can be shown that the four-dimensional θ-angle descends to a worldsheet θ-angle [4, 10]. The end result is that worldsheet complexified FI parameter t is identified with the bulk complexified gauge coupling τ (2.1):

$$t = \tau\ . \tag{2.47}$$

We now turn to the fermionic part of the worldsheet Lagrangian, given by

$$\begin{aligned} L_{\rm fermi} =\ & 2iT\left(\bar{\chi}_-\partial_+\chi_- + \bar{\chi}_+\partial_-\chi_+\right) + 2i\sum_{i=1}^{N_c}\left(\bar{\xi}_{-i}\mathcal{D}_+\xi_{-i} + \bar{\xi}_{+i}\mathcal{D}_-\xi_{+i}\right) \\ & -\sqrt{2}\sum_{i=1}^{N_c}\left(\bar{\sigma}\bar{\xi}_{+i}\xi_{-i} + \sigma\bar{\xi}_{-i}\xi_{+i} + \bar{\phi}_i(\xi_{-i}\zeta_+ - \xi_{+i}\zeta_-) + \phi_i(\bar{\zeta}_-\bar{\xi}_{+i} - \bar{\zeta}_+\bar{\xi}_{-i})\right) \end{aligned} \tag{2.48}$$

[6] This follows by taking the time dependent ansatz for the orientational modes: $\mathcal{D}_t Q_i = \delta Q_i$ and $F_{0z} = \delta A_z$, with $\delta\phi_i = \dot{\phi}_i$. Inserting this into the four dimensional kinetic terms gives,

$$\int dx^1 dx^2 \left(\frac{1}{2e^2}F_{0i}^2 + |\mathcal{D}_t Q_i|^2\right) = \int dx^1 dx^2 \left(\frac{1}{e^2}|\mathcal{D}_i q|^2 + v^2(q-1)^2(q+1)^2\right)\frac{|\mathcal{D}_t\phi_i|^2}{r} = \frac{2\pi}{e^2}\frac{|\mathcal{D}_t\phi_i|^2}{r}$$

where the integral is recognized as the same one that appears in computing the vortex tension.

The fermions $\xi_{\pm i}$ both have charge +1 under the U(1) gauge symmetry, which is now seen to implement the full identification (2.37): $\phi_i \to e^{i\alpha}\phi_i$ and $\xi_\pm \to e^{i\alpha}\xi_\pm$. The vector multiplet fermions $\zeta_\pm$ have no kinetic term and act as Grassmannian Lagrange multipliers, imposing the constraint (2.38): $\sum_i \bar{\phi}_i \xi_{\pm i} = 0$. Finally, the role of σ is to mediate a four-fermi interaction for the super-orientation modes. Upon integrating out σ, we have

$$\mathcal{L}_{4-\text{fermi}} = -2|\sigma|^2|\phi_i|^2 - \sqrt{2}\bar{\sigma}\bar{\xi}_{+i}\xi_{-i} - \sqrt{2}\sigma\bar{\xi}_{-i}\xi_{+i} = -\frac{|\bar{\xi}_{-i}\xi_{+i}|^2}{r}. \tag{2.49}$$

Four-fermi terms of this kind are typical for soliton dynamics in supersymmetric theories. We pause here to review how they arise. In deriving the Dirac equations (2.26) and (2.27) we set $A = \tilde{Q}_i = 0$. This is valid in the background of the bosonic vortex. However, it is no longer true in the presence of fermions since fermi bilinears act as a source for these fields. For example, the Yukawa couplings involving $A^\dagger$ contribute to the equation of motion,

$$\mathcal{D}^2 A + i\sqrt{2}[\lambda, \eta] - \sqrt{2}e^2\bar{\tilde{\psi}}_i\tilde{\psi}_i + e^2\{Q_iQ_i^\dagger + \tilde{Q}_i^\dagger\tilde{Q}_i, A\} = 0 \tag{2.50}$$

The solution to this equation then feeds back into the Dirac equations (2.25) and must be solved iteratively, order by order in the number of Grassmannian collective coordinates. This is a finite, but somewhat complicated procedure (see [17] for a simple quantum mechanical model where it may be carried through to completion). Thankfully, the end result (2.49) is dictated by supersymmetry.

2.4.1 Symmetries

The four-dimensional $\mathcal{N} = 2$ theory has two U(1) R-symmetries[7] that we will call $U(1)_R$ and $U(1)_V$. The charges of the various fields under $U(1)_R \times U(1)_V$ are listed in the table.

	A	λ	η	Q	$\tilde{Q}$	ψ	$\tilde{\psi}$
$U(1)_R$	2	1	1	0	0	-1	-1
$U(1)_V$	0	1	-1	1	1	0	0 .

Both of these symmetries descend to the vortex worldsheet, where they appear as the two R-symmetries of the $\mathcal{N} = (2,2)$ superalgebra. The action on the fermionic collective coordinates of the vortex can can be read directly from (2.33). We have

	σ	ζ_+	ζ_-	ϕ	ξ_+	ξ_-	z	χ_+	χ_-
$U(1)_R$	2	-1	1	0	-1	1	0	-1	1
$U(1)_V$	0	1	1	0	-1	-1	0	-1	-1
$U(1)_Z$	0	1	1	0	-1	-1	2	1	1 .

The $U(1)_R$ symmetry is axial; it suffers an anomaly in the quantum theory of the vortex (as, indeed, does the $U(1)_R$ in four-dimensions). In contrast $U(1)_V$ is a vector R-symmetry on the worldsheet.

The vortex theory also includes a further global $U(1)_Z$ symmetry, which arises from rotating the vortex string in the $z = x^1 + ix^2$ plane. The charges of the worldsheet fields

[7] In the absence of the FI parameter v^2, the theory has an $SU(2)_R$ symmetry, under which (λ, η) and $(Q, \tilde{Q}^\dagger)$ both transform as doublets. The FI parameter breaks $SU(2)_R \to U(1)_V$.

under $\mathrm{U}(1)_Z$ are listed in the table and follow from (2.34) and (2.36). There exists a suitable linear combination of $\mathrm{U}(1)_Z$ and $\mathrm{U}(1)_V$ which simply rotates the phase of the chiral multiplet Z, leaving all other fields invariant.

There are other, translational, symmetries of the worldsheet theory that reflect the fact that z and $\chi_\pm$ are all Goldstone modes, arising from broken translation and supersymmetry invariance respectively. In both cases, this ensures they have only derivative couplings. In particular, it is the existence of these symmetries that prevents the Goldstino modes $\chi_\pm$ from appearing in the four-fermi term (2.49).

2.5 $\mathcal{N} = 2$ preserving deformations

So far we have described vortices in only the simplest $\mathcal{N} = 2$ theory with $N_f = N_c$. There are a number of ways to deform and augment our theory that preserve $\mathcal{N} = 2$ supersymmetry. Here we list them and describe their effect on the worldsheet. We postpone until section 4 a discussion of deformations that break the four dimensional supersymmetry to $\mathcal{N} = 1$.

2.5.1 Adding masses

The simplest deformation of our theory that preserves $\mathcal{N} = 2$ supersymmetry is to add a complex mass parameter m_i for each hypermultiplet. The superpotential (2.2) now becomes

$$\mathcal{W}_{\mathcal{N}=2} = \sqrt{2}\sum_{i=1}^{N_f} \tilde{Q}_i(A - m_i)Q_i. \tag{2.51}$$

The vacuum (2.4) survives only if we turn on the adjoint scalar field A to cancel the F-term contributions,

$$Q^a{}_i = v\delta^a{}_i \quad , \quad \tilde{Q}_i = 0 \quad , \quad A = \mathrm{diag}(m_1, \ldots, m_{N_c}). \tag{2.52}$$

The vortex moduli space does not fare well under this deformation. It can be simply shown that the masses m_i lift the internal $\mathbf{CP}^{N_c-1}$ vortex moduli space, leaving behind N_c distinct, isolated vortex solutions, each of which carries magnetic flux in a different diagonal $\mathrm{U}(1)$ subgroup of the $\mathrm{U}(N_c)$ gauge group, supported by a different Q_i winding at infinity.

It was shown in [18, 3, 4] that the 4d masses m_i induce “twisted masses” [19] for the fields on the vortex worldsheet. In the language of $\mathcal{N} = (2, 2)$ superfields, this deformation replaces the standard kinetic terms for Φ_i by

$$\sum_{i=1}^{N_c} \bar{\Phi}_i e^{2U}\Phi_i \longrightarrow \sum_{i=1}^{N_c} \bar{\Phi}_i \exp\left(2U - 2\theta^-\bar{\theta}^+ m_i - 2\theta^+\bar{\theta}^- m_i^\dagger\right)\Phi_i. \tag{2.53}$$

In terms of components, the vortex theory (neglecting for now the Z multiplet whose dynamics remains unchanged) becomes,

$$\begin{aligned}\mathcal{L}_{\rm vortex} = &\sum_{i=1}^{N_c}\left(|\mathcal{D}_m\phi_i|^2+|F_i|^2-2|\sigma-m_i|^2|\phi_i|^2\right)+D\left(\sum_{i=1}^{N_c}|\phi_i|^2-r\right)+\frac{\theta}{2\pi}u_{01}\\ &+\sum_{i=1}^{N_c}2i\left(\bar{\xi}_{-i}\mathcal{D}_+\xi_{-i}+\bar{\xi}_{+i}\mathcal{D}_-\xi_{+i}\right)-\sqrt{2}(\bar{\sigma}-\bar{m}_i)\bar{\xi}_{+i}\xi_{-i}-\sqrt{2}(\sigma-m_i)\bar{\xi}_{-i}\xi_{+i}\\ &-i\sqrt{2}\sum_{i=1}^{N_c}\Big(\bar{\phi}_i(\xi_{-i}\zeta_+-\xi_{+i}\zeta_-)+\phi_i(\bar{\zeta}_-\bar{\xi}_{+i}-\bar{\zeta}_+\bar{\xi}_{-i})\Big).\end{aligned} \tag{2.54}$$

Note that the masses m_i break the $\mathrm{U}(1)_R$ symmetry, both in four-dimensions and on the vortex worldsheet. The twisted masses on the worldsheet have the desired effect of lifting the $\mathbf{CP}^{N_c-1}$ moduli space of the vortex theory, leaving behind N_c isolated vacua given by

$$|\phi_i|^2 = r\delta_{ij} \quad , \quad \sigma = m_j \quad j=1,\ldots,N_c. \tag{2.55}$$

These different vacua of the worldsheet theory are identified with the different vortex solutions in four-dimensions. Kinks interpolating between these vacua on the worldsheet correspond to magnetic monopoles in four-dimensions, confined to lie on the vortex string by the Meissner effect [18].

2.5.2 Adding flavors

We now consider the theory with $N_f > N_c$ fundamental hypermultiplets. The D-term and F-term vacuum conditions in four-dimensions read

$$\sum_{i=1}^{N_f} Q_i Q_i^\dagger - \tilde{Q}_i^\dagger \tilde{Q}_i = v^2 \quad \text{and} \quad \sum_{i=1}^{N_f} Q_i\tilde{Q}_i = 0. \tag{2.56}$$

When $m_i = 0$, there are no further conditions, and there is a $2N_c(N_f-N_c)$ dimensional Higgs branch of the theory. For the purposes of this section, we place ourselves in the particular vacuum $\tilde{Q}_i = 0$ and

$$Q^a_{\ i} = v\delta^a_{\ i} \quad a,i=1,\ldots N_c \tag{2.57}$$

with $Q_i = 0$ for $i = N_c+1,\ldots,N_f$. This is to be supplemented by $A = 0$. If we now turn on masses for the hypermultiplets, the vacuum (2.57) survives, with $A = \mathrm{diag}(m_1,\ldots,m_{N_c})$.

Vortices in theories with $N_f > N_c$ have a rather different character than those in the $N_f = N_c$ theory. The most noticeable difference is that they gain extra bosonic collective coordinates, among them a scale size. These additional collective coordinates are non-normalizable when $m_i = 0$ [20, 21] but become normalizable when finite masses m_i are turned on for $i = N_c+1,\ldots,N_f$ [22]. Vortices of this kind are sometimes referred to as semi-local vortices: a review of these objects in Abelian theories can be found in [23], while a detailed discussion in non-Abelian theories was given in [22].

An effective dynamics for the vortex worldsheet in theories with $N_f > N_c$ was proposed in [1], based on a D-brane construction. It is once again an $\mathcal{N} = (2,2)$ supersymmetric U(1) gauge theory, now with N_c chiral multiplets Ψ_i of charge +1 and a further $(N_f - N_c)$ chiral multiplets $\tilde{\Psi}_j$ of charge −1. The D-term for this theory reads

$$D = \sum_{i=1}^{N_c} |\phi_i|^2 - \sum_{j=1}^{N_f - N_f} |\tilde{\phi}_j|^2 - r = 0 \tag{2.58}$$

which, together with the gauge action $\phi_i \to e^{i\alpha}\phi_i$ and $\tilde{\phi}_i \to e^{-i\alpha}\tilde{\phi}_i$, defines the Higgs branch of the vortex theory. This Higgs branch is conjectured to coincide the vortex moduli space. As in the previous section, assigning complex masses m_i to the four dimensional hypermultiplets induces twisted masses m_i, $i = 1, \ldots, N_c$ for the Φ_i fields, and twisted masses $\tilde{m}_j = m_{j+N_c}$, $j = 1, \ldots, N_f - N_c$ for $\tilde{\Phi}_j$.

The presence of the negatively charged fields $\tilde{\phi}_j$ means that the moduli space (2.58) is now non-compact, corresponding to the scaling mode of the vortex. Note however that the natural metric on the Higgs branch does not coincide with the natural metric on the vortex moduli space. In particular, the non-normalizability of the scaling modes as $m_i \to 0$ is not reproduced in this model. Nonetheless, it has been shown that the vortex theory (2.58) does indeed correctly capture the quantum dynamics of the vortex string [4, 7].

Higgs expectation values. When $N_f > N_c$ and $m_i = 0$, the vacuum conditions (2.56) in the four-dimensional theory have a moduli space of solutions. We may ask what happens to the vortex string as we change the expectation values of Q_i and $\tilde{Q}_i$ such that (2.56) remains satisfied. The answer to this question was given in [7]: turning on expectation values for $\tilde{Q}_i$ induces a superpotential on the vortex string worldsheet. For completeness, we briefly describe this deformation here.

First some notation: define the gauge invariant meson operator

$$M_i^j \equiv \tilde{Q}^j Q_i. \tag{2.59}$$

It is not hard to show that in four-dimensional vacua for which $M \neq 0$, the space of BPS vortex solutions is greatly reduced. The key point is that a vacuum expectation value for $\tilde{Q}$ does not allow a BPS vortex to live in the associated part of the gauge group. This follows from the mathematical fact that there is no holomorphic line bundle of negative degree. In a more physical language, a direct analysis of the vortex equations reveals that BPS vortices do not exist in U(1) theories if both negatively and positively charged fields gain an expectation value [24, 25]. The upshot of this is that the vortex moduli space is partly lifted in four dimensional vacua for which $M \neq 0$. It was shown in [7] that this effect is captured on the vortex worldsheet by the introduction of a superpotential of the form,

$$\mathcal{W}_{(2,2)} \sim \sum_{i=1}^{N_f} \sum_{j=1}^{N_f - N_c} M_i^{j+N_c} \, \tilde{\Phi}_j \Phi^i. \tag{2.60}$$

2.6 Multiple vortices

So far we have focussed on the theory for a single vortex string for which, at least in the case $N_f = N_c$, symmetries are sufficient to dictate the dynamics. In [1], a D-brane construction was used to derive a worldsheet theory which describes the interactions of $k > 1$ parallel vortex strings. The D-brane construction starts with the usual Hanany-Witten set-up for $\mathcal{N} = 2$ four-dimensional gauge theories [26, 27], consisting of D4-branes attached to parallel NS5-branes. Separating the NS5-branes in the direction out of the page induces the FI parameter v^2. The vortex strings arise as stretched D2-branes as shown in figure 2. The worldvolume theory of k vortex strings is given by an $\mathcal{N} = (2,2)$ U(k) non-Abelian gauge theory with matter content,

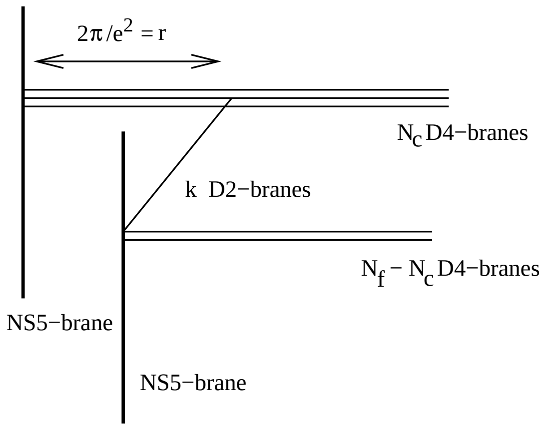

Figure 2: The brane configuration.

$$\begin{aligned}\mathrm{U}(k)\ \text{Vector Multiplet}\ U\ &+\ \text{Adjoint Chiral Multiplet}\ Z\\ &+ N_c\ \text{Fundamental Chiral Multiplets}\ \Phi_i\\ &+ N_f - N_c\ \text{Anti-Fundamental Chiral Multiplets}\ \tilde{\Phi}_j.\end{aligned}$$

The complexified worldsheet FI parameter is again equated to the 4d complexified gauge coupling, $t = \tau$, or

$$ir + \frac{\theta_{2d}}{2\pi} = \frac{2\pi i}{e^2} + \frac{\theta_{4d}}{2\pi}. \tag{2.61}$$

The D-term condition for the worldsheet theory is now $u(k)$ valued and is given by,

$$\sum_{i=1}^{N_c} \phi_i \phi_i^\dagger - \sum_{j=1}^{N_f - N_c} \tilde{\phi}_j^\dagger \tilde{\phi}_j + T[z, z^\dagger] = r\, 1_k. \tag{2.62}$$

This provides k^2 constraints on the $2k(N_f + k)$ degrees of freedom in ϕ_i, $\tilde{\phi}_j$ and z. After dividing by U(k) gauge transformations, we are left with a $2kN_f$ dimensional manifold which defines the target space for the vortex string sigma-model. This was conjectured in [1] to coincide with $2kN_f$ dimensional vortex moduli space. This quotient construction has subsequently been derived from a direct analysis of the non-Abelian vortex equations [28, 29].

The $4kN_c$ fermionic zero modes of k parallel vortex strings live in the U(k) adjoint valued $\chi_\pm$ and the fundamental $\xi_{\pm i}$, subject to the $2k^2$ complex constraints arising from the auxiliary fermions $\zeta_\pm$,

$$\sum_i \phi_i \bar{\xi}_{\pm i} + [z, \bar{\chi}_\pm] = 0. \tag{2.63}$$

In the case of a single $k = 1$ vortex, these reduce to the constraints (2.38).

Note that the vacuum moduli space (2.62) inherits a metric from the canonical kinetic terms for ϕ_i and z. This metric is known not to agree with the standard Manton metric [30, 31] on the vortex moduli space (except in the special case $k = 1$ and $N_f = N_c$ that we described in detail earlier). This is because the limit in which the $d = 1 + 1$ gauge theory on the D2-branes decouples from other stringy modes is different from the limit in which the D2-branes are described as vortices in the $d = 3 + 1$ dimensional theory on the D4-branes; the two descriptions hold in different regimes of validity as we vary the parameters of the brane set-up. Nonetheless, if one is interested in computing objects protected by supersymmetry — such as the classical, or quantum, masses of BPS states in the vortex theory —- it should be valid to work with the gauge linear sigma model. In practice, this claim has been confirmed only for the $k = 1$ theory with $N_f > N_c$. It has also been confirmed that the intricate topology of the $k = 2$ vortex string moduli space in the $N_f = N_c = 2$ is correctly captured by the gauged linear sigma model [32−34].

3. $\mathcal{N} = (0, 2)$ supersymmetry

In the previous section we have worked with both $\mathcal{N} = 1$ superfields in four dimensions, A, Q_i and $\tilde{Q}_j$, as well as $\mathcal{N} = (2, 2)$ superfields in two dimensions, Σ, Φ_i and $\tilde{\Phi}_j$. From now on we will deal with $\mathcal{N} = (0, 2)$ supersymmetry in two dimensions. Since this may be less familiar to some readers we devote this section to a review of the structure of $\mathcal{N} = (0, 2)$ superfields [35, 36] and their relationship to $\mathcal{N} = (2, 2)$ theories. The presentation follows [37] and [38].

3.1 Superfields

$\mathcal{N} = (0, 2)$ supersymmetry is generated by two right-moving, and no left-moving, supersymmetries. The two chiral supercharges are Q_+ and $\bar{Q}_+$. The $(0, 2)$ superspace is parameterized by the bosonic coordinates $y^\pm = (y^0 \pm y^1)$ and their fermionic partners θ^+ and $\bar{\theta}^+$. The action of the supersymmetry generators in superspace is given as

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+(\partial_0 + \partial_1) \\ \bar{Q}_+ &= -\frac{\partial}{\partial\bar{\theta}^+} - i\theta^+(\partial_0 + \partial_1). \end{aligned} \tag{3.1}$$

These commute with the superderivatives,

$$\begin{aligned} D_+ &= \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+(\partial_0 + \partial_1) \\ \bar{D}_+ &= -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+(\partial_0 + \partial_1) \end{aligned} \tag{3.2}$$

which satisfy $\{D_+, D_+\} = \{\bar{D}_+ \bar{D}_+\} = 0$ and $\{D_+, \bar{D}_+\} = 2i\partial_+$. We now describe the different superfields of interest.

Gauge multiplets. We start with the real, adjoint valued, gauge multiplet U, which has the component expansion

$$U = (u_0 - u_1) - 2i\theta^+\bar{\zeta}_- - 2i\bar{\theta}^+\zeta_- + 2\theta^+\bar{\theta}^+ D. \tag{3.3}$$

Already we see the chiral nature of the supersymmetry, since only the combination $u_- = u_0 - u_1$ of the two-dimensional gauge field appears in the superfield, together with a left moving fermion ζ_-. The scalar field D will be seen to be auxiliary. The covariant superderivatives are given by

$$\begin{aligned}\mathcal{D}_+ &= \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+(\mathcal{D}_0 + \mathcal{D}_1)\\ \bar{\mathcal{D}}_+ &= -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+(\mathcal{D}_0 + \mathcal{D}_1)\end{aligned}$$

where

$$\mathcal{D}_0 + \mathcal{D}_1 = \partial_0 + \partial_1 - i(u_0 + u_1)$$

includes the u_+ component of the gauge field, but no fermions. Meanwhile, the gauginos are included in the remaining covariant superderivative,

$$\mathcal{D}_0 - \mathcal{D}_1 = \partial_0 - \partial_1 - iU. \tag{3.4}$$

The field strength lives naturally in a fermi multiplet (which we shall define shortly) given by the usual commutator of derivatives:

$$\Upsilon = [\bar{\mathcal{D}}_+, \mathcal{D}_0 - \mathcal{D}_1] = -2\Big(\zeta_- - i\theta^+(D - iv_{01}) - i\theta^+\bar{\theta}^+(\mathcal{D}_0 + \mathcal{D}_1)\zeta_-\Big). \tag{3.5}$$

Here the field strength is $u_{01} = \partial_0 u_1 - \partial_1 u_0 - i[u_0, u_1]$. The kinetic terms for the gauge multiplet are then given by integration over all of superspace $d^2\theta = d\theta^+\, d\bar{\theta}^+$,

$$\begin{aligned}S_{\text{gauge}} &= \frac{1}{8g^2}\,\text{Tr}\int d^2y\, d^2\theta\ \Upsilon^\dagger\Upsilon\\ &= \frac{1}{g^2}\,\text{Tr}\int d^2y\,\left(\frac{1}{2}u_{01}^2 + i\bar{\zeta}_-(\mathcal{D}_0 + \mathcal{D}_1)\zeta_- + D^2\right)\end{aligned} \tag{3.6}$$

but, in fact, will not be required in the following.

Chiral multiplets. The chiral multiplets of $(0,2)$ theories are bosonic superfields Φ, living in any representation R of the gauge group. They satisfy

$$\bar{\mathcal{D}}_+\Phi = 0. \tag{3.7}$$

Chiral multiplets contain right-moving fermions ξ_+, paired with a complex boson ϕ. Their component expansion gives

$$\Phi = \phi + \sqrt{2}\theta^+\xi_+ - i\theta^+\bar{\theta}^+(D_0 + D_1)\phi \tag{3.8}$$

where $(D_0 + D_1)$ is now the usual bosonic covariant derivative. The kinetic terms for the chiral multiplet are given by the action,

$$\begin{aligned} S_{\text{chiral}} &= -\frac{i}{2}\int d^2y\, d^2\theta\ \bar{\Phi}(\mathcal{D}_0 - \mathcal{D}_1)\Phi \qquad (3.9)\\ &= \int d^2y\ \left(-|D_\alpha\phi|^2 + i\bar{\xi}_+(D_0 - D_1)\xi_+ - i\sqrt{2}\bar{\phi}\zeta_-\xi_+ + i\sqrt{2}\bar{\xi}_+\bar{\zeta}_-\phi + \bar{\phi}D\phi\right). \end{aligned}$$

The scalar field ϕ couples to the auxiliary field D, to give rise to the usual D-term (Note that for Abelian theories, if Φ has charge p then one should replace $\zeta_- \to p\zeta_-$ and $D \to pD$ in the above action.).

Fermi multiplets. One novel feature of $(0,2)$ theories that is not shared by the non-chiral $(2,2)$ theories is the existence of a fermionic multiplet Γ, containing only left moving fermions χ_- and no propagating bosons. Like the chiral multiplets, they can live in any representation R of the gauge group. The fermi multiplet satisfies

$$\bar{\mathcal{D}}_+\Gamma = \sqrt{2}E \qquad (3.10)$$

where $\bar{\mathcal{D}}_+E = 0$, which can be solved by taking E to be a holomorphic function of chiral superfields $E = E(\Phi_i)$. The fermi multiplet has component expansion

$$\Gamma = \chi_- - \sqrt{2}\theta^+G - i\theta^+\bar{\theta}^+(D_0 + D_1)\chi_- - \sqrt{2}\bar{\theta}^+E. \qquad (3.11)$$

Note that the superfield Υ containing the field strength is of this type, with $\bar{\mathcal{D}}_+\Upsilon = 0$. In general, E itself will also have a θ expansion,

$$E(\Phi_i) = E(\phi_i) + \sqrt{2}\theta^+\frac{\partial E}{\partial\phi_i}\xi_{+i} - i\theta^+\bar{\theta}^+(D_0 + D_1)E(\phi_i) \qquad (3.12)$$

The kinetic terms for the fermi multiplet are

$$\begin{aligned} S_{\text{fermi}} &= -\frac{1}{2}\int d^2y\, d^2\theta\ \bar{\Gamma}\Gamma \qquad (3.13)\\ &= \left(i\bar{\chi}_-(D_0 + D_1)\chi_- + |G|^2 - |E(\phi_i)|^2 - \bar{\chi}_-\frac{\partial E}{\partial\phi^i}\xi_{+i} + \bar{\xi}_{+i}\frac{\partial\bar{E}}{\partial\bar{\phi}_i}\chi_-\right) \end{aligned}$$

We see that the complex scalar G is an auxiliary field, lacking a kinetic term. Also note that the function $E(\phi)$ appears as a potential term in the Lagrangian.

3.2 Superpotentials

In $\mathcal{N} = (0,2)$ theories the auxiliary field G lives in a fermi multiplet Γ, rather than a chiral multiplet. A superpotential $J(\Phi_i)$ is a holomorphic function of chiral superfields and a suitable action may be constructed by integrating terms of the form ΓJ over half of superspace. Most generally we can introduce a superpotential J^a for each fermi multiplet Γ^a,

$$\begin{aligned} S_J &= -\frac{1}{\sqrt{2}}\sum_a\int d^2y\, d\theta^+\ \ \Gamma_a\ J^a(\Phi_i)|_{\bar{\theta}^+=0} + \text{ h.c.}\\ &= \sum_a\int d^2y\ \ G_aJ^a(\phi_i) + \sum_i\chi_{-a}\frac{\partial J^a}{\partial\phi_i}\xi_{+i} + \text{ h.c.}\ . \qquad (3.14) \end{aligned}$$

This integration over half of superspace yields an $\mathcal{N} = (0,2)$ supersymmetric invariant action if and only if $\bar{D}_+(\Gamma_a J^a) = 0$, which requires

$$\sum_a E_a J^a = 0. \tag{3.15}$$

Of course, the combination $\Gamma_a J^a$ is also required to be gauge invariant. An important example of the superpotential is the Fayet-Iliopoulos and theta term which are packaged in the complex combination $t = ir + \theta/2\pi$. The interaction can be written as

$$\begin{aligned} S_{D\theta} &= \frac{t}{4}\,\mathrm{Tr}\int d^2y d\theta^+\ \Upsilon|_{\bar{\theta}^+=0} + \text{ h.c.} \\ &= \mathrm{Tr}\int d^2y\Big(-rD + \frac{\theta}{2\pi}u_{01}\Big). \end{aligned} \tag{3.16}$$

3.3 $\mathcal{N} = (2,2)$ decomposition

It will prove useful for orientation to recall how the more familiar $\mathcal{N} = (2,2)$ superfields decompose into their $\mathcal{N} = (0,2)$ counterparts. The conventions below are taken from [37].

One can enlarge $\mathcal{N} = (0,2)$ superspace to $\mathcal{N} = (2,2)$ superspace through the addition of two further fermionic components θ^- and $\bar{\theta}^-$. The corresponding superderivatives are

$$D_- = \frac{\partial}{\partial\theta^-} - i\bar{\theta}^-(\partial_0 - \partial_1) \quad , \quad \bar{D}_- = -\frac{\partial}{\partial\bar{\theta}^-} + i\theta^-(\partial_0 - \partial_1). \tag{3.17}$$

The $\mathcal{N} = (2,2)$ vector multiplet $V_{(2,2)}$ decomposes into an $\mathcal{N} = (0,2)$ vector multiplet V described in (3.3), together with an $\mathcal{N} = (0,2)$ chiral multiplet Σ. This chiral multiplet inherits the right moving fermion ζ_+ and the complex scalar field σ contained in $V_{(2,2)}$. It is most simply described by reduction from the $\mathcal{N} = (2,2)$ twisted chiral multiplet containing the field strength $\Sigma_{(2,2)} = (1/\sqrt{2})\{\bar{\mathcal{D}}_+, \mathcal{D}_-\}$, in terms of which the $\mathcal{N} = (0,2)$ chiral multiplet is given by

$$\Sigma = \Sigma_{(2,2)}\big|_{\theta^-=\bar{\theta}_-=0}\,. \tag{3.18}$$

An $\mathcal{N} = (2,2)$ chiral multiplet $\Phi_{(2,2)}$ satisfies $\bar{D}_+\Phi_{(2,2)} = \bar{D}_-\Phi_{(2,2)} = 0$. This chiral multiplet decomposes into an $\mathcal{N} = (0,2)$ chiral multiplet Φ and a fermi multiplet Γ, defined by

$$\begin{aligned} \Phi &= \Phi_{(2,2)}\big|_{\theta^-=\bar{\theta}^-=0} \\ \Gamma &= \frac{1}{\sqrt{2}}\mathcal{D}_-\ \Phi_{(2,2)}\big|_{\theta^-=\bar{\theta}^-=0}\,. \end{aligned} \tag{3.19}$$

If $\Phi_{(2,2)}$ transforms under a representation R of the gauge group, then both Φ and Γ also transform under R. A quick computation yields $\bar{\mathcal{D}}_+\Gamma = 2i\Sigma\Phi$, meaning that, in the notation of (3.10), $\mathcal{N} = (2,2)$ supersymmetry imposes,

$$E = i\sqrt{2}\Sigma\Phi. \tag{3.20}$$

The final $\mathcal{N} = (2,2)$ multiplet of interest is a twisted chiral multiplet $\Sigma_{(2,2)}$, satisfying $\bar{D}_+\Sigma_{(2,2)} = D_-\Sigma_{(2,2)} = 0$. Like the $\mathcal{N} = (2,2)$ chiral multiplet, this too decomposes into an $\mathcal{N} = (0,2)$ chiral multiplet Σ and a fermi multiplet F. They are given by,

$$\begin{aligned} \Sigma &= \Sigma_{(2,2)}\big|_{\theta^-=\bar{\theta}^-=0} \\ F &= -\frac{1}{\sqrt{2}}\bar{D}_-\ \Sigma_{(2,2)}\big|_{\theta^-=\bar{\theta}^-=0}\,. \end{aligned} \tag{3.21}$$

Note, however, that from the expansion (2.42), the θ^+ component of the $\mathcal{N} = (0,2)$ chiral multiplet Σ contains the barred fermion, rather than the unbarred fermion,

$$\Sigma = \sigma - i\sqrt{2}\theta^+\bar{\zeta}_+ - i2\theta^+\bar{\theta}^+\partial_+\sigma. \tag{3.22}$$

This subtlety will prove important in what follows. Since twisted chiral multiplets $\Sigma_{(2,2)}$ are always uncharged under the gauge group, the corresponding fermi multiplet satisfies $\bar{\mathcal{D}}_+F = 0$.

3.3.1 The vortex theory in $\mathcal{N} = (0,2)$ language

Let us finish this section by describing the $\mathcal{N} = (2,2)$ vortex theory of section 2 in the language of $\mathcal{N} = (0,2)$ superfields. This will serve to fix notation for what is to come. We decompose the fields as

$$\begin{aligned} \mathcal{N} = (2,2)\ \mathrm{U}(k)\ \text{Vector Multiplet} &\longrightarrow \mathrm{U}(k)\ \text{Vector Multiplet, } U \\ &\qquad + \text{Adjoint Chiral Multiplet } \Sigma \\ \mathcal{N} = (2,2)\ \text{Adjoint Chiral Multiplet} &\longrightarrow \text{Adjoint Chiral Multiplet, } Z \\ &\qquad + \text{Adjoint Fermi Multiplet } \Xi \\ \mathcal{N} = (2,2)\ \text{Fund. Chiral Multiplets} &\longrightarrow \text{Fund. Chiral Multiplets, } \Phi_i \\ &\qquad + \text{Fund. Fermi Multiplets } \Gamma_i \\ \mathcal{N} = (2,2)\ \text{Anti-Fund. Chiral Multiplet} &\longrightarrow \text{Anti-Fund. Chiral Multiplets, } \tilde{\Phi}_j \\ &\qquad + \text{Anti-Fund. Fermi Multiplet } \tilde{\Gamma}_j \end{aligned}$$

where all the objects on the right are $\mathcal{N} = (0,2)$ superfields. As before, $i = 1,\ldots,N_c$ for Φ_i and $j = 1,\ldots N_f - N_c$ for $\tilde{\Phi}_j$. Appendix A contains a list of the different component fields which appear in each of these multiplets.

The $\mathcal{N} = (2,2)$ supersymmetry imposes the relations,

$$\bar{\mathcal{D}}_+\Xi = 2i[\Sigma, Z] \quad , \quad \bar{\mathcal{D}}_+\Gamma_i = 2i(\Sigma - m_i)\Phi_i \quad , \quad \bar{\mathcal{D}}_+\tilde{\Gamma}^j = -2i(\Sigma - \tilde{m}_j)\tilde{\Phi}_j \tag{3.23}$$

(There is no sum over i and j on the right-hand side of these equations). As we have seen, the right-hand side of each of these equations appears as a potential "$|E|^2$" arising in equation (3.13). A further contribution to the worldsheet scalar potential arises from the D-term, which provides the constraint (2.62).

4. The $\mathcal{N} = (0, 2)$ dynamics of vortex strings

It is now time to present new results for the dynamics of vortex strings in theories with $\mathcal{N} = 1$ supersymmetry. Most of this section is devoted to the discussion of a simple deformation of the $\mathcal{N} = 2$ theory by the addition of a superpotential. In section 4.7 we discuss a second class of deformations.

4.1 Adding a superpotential

We start by considering a "Dijkgraaf-Vafa"-like deformation [39], breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$ through the addition of a superpotential for the adjoint superfield A. The superpotential now reads

$$\mathcal{W} = \sqrt{2} \sum_{i=1}^{N_f} \tilde{Q}_i (A - m_i) Q_i + \hat{\mathcal{W}}(A) \tag{4.1}$$

which gives rise to the scalar potential

$$\begin{aligned} V_{4d} = & \frac{e^2}{2} \mathrm{Tr} \left(\sum_{i=1}^{N_f} Q_i Q_i^\dagger - \tilde{Q}_i \tilde{Q}_i^\dagger - v^2 \mathbf{1}_{N_c} \right)^2 + e^2 \mathrm{Tr} | \sum_{i=1}^{N_f} \tilde{Q}_i Q_i - \partial \hat{W} / \partial A |^2 \\ & + \sum_{i=1}^{N_f} \left(Q_i^\dagger \{ A - m_i, \bar{A} - \bar{m}_i \} Q_i + \tilde{Q}_i \{ A - m_i, \bar{A} - m_i \} \tilde{Q}_i^\dagger \right) + \frac{1}{2e^2} \mathrm{Tr} | [A, A^\dagger] |^2 . \end{aligned} \tag{4.2}$$

Let's look at how this superpotential affects the vacuum structure. If $\hat{\mathcal{W}}$ is linear in A then there is merely a constant piece in the F-term above and the Lagrangian still preserves $\mathcal{N} = 2$ supersymmetry. We can perform an $\mathrm{SU}(2)_R$ rotation of the scalar fields $(Q_i, \tilde{Q}_i^\dagger)$ to bring the Lagrangian back to the form (2.3). We will assume that $\hat{\mathcal{W}}$ does not contain a linear piece. In this case, for a generic superpotential $\hat{\mathcal{W}}(A)$, $\tilde{Q}_i$ must turn on in the vacuum. Without loss of generality, we choose the vacuum to be of the form,

$$Q^a{}_i = p_i \, \delta^a_i \quad , \quad \tilde{Q}^a{}_i = \tilde{p}_i \, \delta^a_i \quad , \quad A = \mathrm{diag}(m_1, \dots, m_{N_c}) \tag{4.3}$$

with

$$|p_i|^2 - |\tilde{p}_i|^2 = v^2 \quad \text{and} \quad \tilde{p}_i p_i = \left. \frac{\partial \hat{\mathcal{W}}}{\partial a} \right|_{m_i} \qquad \text{for each } i = 1, \dots, N_c \tag{4.4}$$

4.2 What becomes of the vortex?

Our goal is to understand how this deformation affects the dynamics of the vortex string.[8] Let us firstly consider the case with distinct masses m_i. Before adding the superpotential $\hat{\mathcal{W}}$ there were N_c different BPS vortices, each living in a different $\mathrm{U}(1) \subset \mathrm{U}(N_c)$ and each with a different Q_i, $i = 1, \dots, N_c$ carrying the asymptotic winding. What changes in the presence of $\hat{\mathcal{W}}$?

[8] Vortices in a similar system were studied in [11], but in the limit with $v^2 = 0$, so that the vortex is built around a linear piece of $\hat{\mathcal{W}}$. This gives rise to somewhat different physics from that considered here.

The crucial point to note is that something rather special happens when the superpotential is tuned so that a critical point coincides with one of the masses, say m_k for some $k = 1, \ldots, N_c$

$$\left.\frac{\partial \hat{\mathcal{W}}(a)}{\partial a}\right|_{a=m_k} = 0. \tag{4.5}$$

If this is case, the vacuum equation (4.3) sets $\tilde{Q}_k = 0$. There is then no obstacle in constructing the k^{th} vortex in which Q_k winds; indeed the $\mathcal{N} = 2$ vortex solution remains a solution in the deformed theory.

Vortices of this type in $\mathcal{N} = 1$ theories are often called D-term vortices (the name arises because the symmetry breaking is induced by a FI parameter, or D-term). It was shown in [40] that such objects are 1/2 BPS, preserving two of the four supercharges of the four-dimensional $\mathcal{N} = 1$ theory. In two dimensions, there are two distinct superalgebras with two supercharges: the non-chiral $(1, 1)$ algebra, and the chiral $(0, 2)$ algebra. Given that the previous section was devoted to a review of $\mathcal{N} = (0, 2)$ theories, the reader may guess this will be relevant for the vortex string. Let's now see that this is indeed the case [40]. The $\mathcal{N} = 1$ supersymmetry transformations for the vector multiplet fields are,

$$\begin{aligned} \delta A_\mu &= -i\bar{\epsilon}\sigma_\mu\lambda + i\bar{\lambda}\sigma_\mu\epsilon \\ \delta D &= \bar{\epsilon}\bar{\sigma}^\mu \mathcal{D}_\mu\lambda + \mathcal{D}_\mu\lambda\bar{\sigma}^\mu\epsilon \\ \delta\lambda &= \tfrac{1}{2}\sigma^{\mu\nu}\epsilon F_{\mu\nu} + i\epsilon D. \end{aligned} \tag{4.6}$$

For each chiral multiplet Q_i, they take the form

$$\begin{aligned} \delta Q_i &= \sqrt{2}\epsilon\psi_i \\ \delta F_i &= i\sqrt{2}\bar{\epsilon}\,\bar{\not{\mathcal{D}}}\psi_i - 2i\bar{\epsilon}\lambda Q_i \\ \delta\psi_i &= \sqrt{2}\epsilon F_i + i\sqrt{2}(\not{\mathcal{D}}Q_i)\bar{\epsilon}. \end{aligned} \tag{4.7}$$

Similar transformations also hold for the chiral multiplets $\tilde{Q}_i$ with the appropriate substitutions. Finally, the supersymmetry transformations for the adjoint chiral multiplet A take the form,

$$\begin{aligned} \delta A &= \sqrt{2}\epsilon\eta \\ \delta F &= i\sqrt{2}\bar{\epsilon}\,\bar{\not{\mathcal{D}}}\eta - 2i\bar{\epsilon}[\lambda, A] \\ \delta\eta &= \sqrt{2}\epsilon F + i\sqrt{2}(\not{\mathcal{D}}A)\bar{\epsilon}. \end{aligned} \tag{4.8}$$

The key point here is that the vortex equations (2.6), together with the requirement that $F_i = F = 0$, provide solutions to $\delta\lambda = \delta\psi_i = \delta\eta = 0$. The latter condition $F = 0$ is trivially satisfied when (4.5) holds, for then $\tilde{Q}_i = 0$, while A remains constant. To see which supersymmetries are preserved in this case, it will suffice to examine the $\delta\psi_i$ transformation. Using (2.28), in the background of a stationary vortex so that $\mathcal{D}_+ = \mathcal{D}_- = 0$, we have

$$\delta\psi_{-i} = -2\sqrt{2}i(\mathcal{D}_z Q_i)\bar{\epsilon}_- = 0 \quad \text{and} \quad \delta\psi_{+i} = 2\sqrt{2}i(\mathcal{D}_{\bar{z}} Q_i)\bar{\epsilon}_+. \tag{4.9}$$

In the background of a vortex, with the scalar field satisfying $\mathcal{D}_z Q_i = 0$, we learn that $\bar{\epsilon}_-$ is the preserved supersymmetry; it descends to provide the supersymmetry variation parameter on the worldsheet. Meanwhile, $\bar{\epsilon}_+$ is the broken supersymmetry which generates a single Goldstino mode on the worldsheet. In our notation (2.34), we have $\bar{\epsilon}_+ = \chi_+/4$. (The χ_- collective coordinate in (2.34) arises from the second supersymmetry transformation of the $\mathcal{N} = 2$ theory. Its fate in our $\mathcal{N} = 1$ theory will be discussed shortly). The spinors $\epsilon_\pm$ have definite, and opposite, chirality on the worldsheet. This is the statement that the worldsheet theory preserves chiral $\mathcal{N} = (0, 2)$ supersymmetry, rather than $\mathcal{N} = (1, 1)$.

We have seen that, in the special case that a critical point of $\hat{\mathcal{W}}$ coincides with a mass (4.5), there exists at least one BPS vortex preserving $\mathcal{N} = (0, 2)$ supersymmetry. But what happens if this is not the case? If (4.5) is not satisfied, then there can be no BPS vortex solutions. To see this, note that (4.4) tells us that $\tilde{Q}_k$ gains an expectation value in the 4d vacuum. This means it cannot now remain constant but, must wind asymptotically to ensure that its kinetic term remains finite. A putative BPS vortex must now satisfy,

$$\mathcal{D}_z Q_i = \mathcal{D}_z \tilde{Q}_i = 0. \tag{4.10}$$

Yet Q_i and $\tilde{Q}_i$ have opposite charges. A standard theorem in mathematics — that a line bundle of negative degree has no non-zero holomorphic section — states that there can only be simultaneous solutions to these equations when either $\tilde{Q}_i = 0$ or $Q_i = 0$. (See, for example, equation (3.43) of [37]). One can reach the same conclusion by noting that A is now also sourced in the vortex background and $\delta\eta \neq 0$.[9] Of course, simple topological arguments imply that vortex strings still exist. However, they must satisfy the full second order equations of motion, rather than the first order Bogomolnyi equations, and their tension is strictly greater than the BPS bound $T = 2\pi v^2$.

4.3 Vortex dynamics

In section 2, we described the $\mathcal{N} = (2, 2)$ U(k) theory on the vortex worldsheet that captures the dynamics of k parallel vortex strings in $\mathcal{N} = 2$ four dimensional gauge theories. We would like to understand how the worldsheet theory reacts to the superpotential $\hat{\mathcal{W}}(A)$, breaking the four dimensional supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. We have seen above that the vortices in the theory with superpotential $\hat{\mathcal{W}}(A)$ are classically BPS, preserving $\mathcal{N} = (0, 2)$ supersymmetry, when equation (4.5) holds; otherwise there are no BPS vortices. We would like to see this from the worldsheet.

In fact, there is a unique deformation on the vortex worldsheet that preserves $\mathcal{N} = (0, 2)$ supersymmetry and reproduces the expected vacuum structure described above. Recall from section 3.2 that superpotentials in $\mathcal{N} = (0, 2)$ theories are constructed from fermi multiplets. The only such multiplet with a suitable transformation under the U(k) gauge symmetry is Ξ, containing χ_- and the complex auxiliary field G_Z. The worldsheet

[9] The lack of BPS vortices in this case is entirely analogous to the statement that F-term vortices are not BPS in $\mathcal{N} = 1$ theories [40]. In our set-up, the value of $\partial\hat{\mathcal{W}}/\partial a$ evaluated at $a = m_k$ plays the role of the constant in the F-term in [40].

deformation is given by the $\mathcal{N}=(0,2)$ superpotential,

$$
\begin{aligned}
S_{\mathcal{W}} &\equiv -\frac{1}{\sqrt{2}}\mathrm{Tr}_k \int d\theta^+ \ \Xi\, J(\Sigma)\Big|_{\bar\theta^+=0} - \text{h.c.} \\
&= -\frac{1}{\sqrt{2}}\mathrm{Tr}_k \int d\theta^+ \ \Xi\, \frac{\partial\hat{\mathcal{W}}(\Sigma)}{\partial\Sigma}\Big|_{\bar\theta^+=0} - \text{h.c.}
\end{aligned}
\tag{4.11}
$$

(up to some overall, unfixed, constant of proportionality). Note that a superpotential of this form is a viable holomorphic term since $\bar{\mathcal{D}}_+\Xi = 2i[\Sigma, Z] \equiv i\sqrt{2}E_\Xi$ and

$$
\mathrm{Tr}\ E_\Xi J = \sqrt{2}\mathrm{Tr}\left([\Sigma, Z]\,\frac{\partial\hat{\mathcal{W}}(\Sigma)}{\partial\Sigma}\right) = 0 \tag{4.12}
$$

which satisfies the requirement (3.15). In principle there could also be σ-dependent deformations of the kinetic terms for Λ_i and Ξ. As is common in supersymmetric field theories, we will have less control over these "D-term" deformations, but will see that the superpotential (4.11) captures much of the important physics.

The deformation (4.11) has implications for both the bosonic and fermionic zero modes of the vortex strings. We defer a discussion of the fermions to the next subsection; we start here by studying the bosonic zero modes. The extra bosonic term on the vortex worldsheet arising from (4.11) is a potential,[10]

$$
V_{2d} = \mathrm{Tr}_k\left(T|G_Z|^2 + G_Z\frac{\partial\hat{\mathcal{W}}(\sigma)}{\partial\sigma} + \text{h.c.}\right) = \frac{1}{T}\mathrm{Tr}_k\left|\frac{\partial\hat{\mathcal{W}}(\sigma)}{\partial\sigma}\right|^2 . \tag{4.13}
$$

We will now show that this gives the expected vacuum structure by studying the $k=1$ vortex theory in some detail; the extension to $k>1$ then follows.

4.3.1 An example: $k=1$ with $N_f = N_c$

To illustrate the role of the superpotential (4.11), let's look at the familiar $k=1$ theory of a single vortex in the case with $N_f = N_c$ flavors. As we discussed in detail in section 2, when $\hat{\mathcal{W}}=0$ the internal moduli space is $\mathbf{CP}^{N_c-1}$ with ϕ_i providing homogeneous coordinates. Once we turn on the superpotential $\hat{\mathcal{W}}$, the bosonic part of the worldsheet theory is given by

$$
\mathcal{L}_{\text{bose}} = T|\partial_m z|^2 + \sum_{i=1}^{N_c}\left(|\mathcal{D}_m\phi_i|^2 - 2|\sigma - m_i|^2|\phi_i|^2\right) + D\left(\sum_{i=1}^{N_c}|\phi_i|^2 - r\right) - \frac{1}{T}\left|\frac{\partial\hat{\mathcal{W}}}{\partial\sigma}\right|^2 + \frac{\theta}{2\pi}u_{01}.
$$

In the presence of distinct, non-zero masses m_i, this worldsheet theory has a supersymmetric ground state (i.e. with vanishing vacuum energy) at

$$
|\phi_j|^2 = r\delta_{ij} \quad , \quad \sigma = m_i \tag{4.14}
$$

[10] A note on dimensions: In 4d, $[\hat{\mathcal{W}}(A)] = 3$, which ensures that the scalar potential has the correct dimensions: $[|\partial\hat{\mathcal{W}}/\partial A|^2] = 4$. In 2d the auxiliary field has dimension $[\sigma]=1$, so that $[\partial\hat{\mathcal{W}}/\partial\sigma] = 2$. The presence of the vortex tension, with $[T]=2$, means that the worldsheet scalar potential (4.13) has the correct scaling for the two dimensional worldsheet.

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only if $\hat{\mathcal{W}}(\sigma)$ has a critical point at $\sigma = m_i$

$$\left.\frac{\partial \hat{\mathcal{W}}(\sigma)}{\partial \sigma}\right|_{\sigma=m_i} = 0. \tag{4.15}$$

This coincides with the expectations of the previous section: BPS vortices only exist when (4.15) holds.

When the masses do not coincide with the critical points, and there are no BPS vortices, the potential $|\partial\hat{\mathcal{W}}/\partial\sigma|^2/T$ determines the vacuum energy of the vortex string. One could try to compare this to the excess tension of the non-BPS vortex string, above the bound $T = 2\pi v^2$, but this unfortunately suffers from the previously mentioned ambiguity in classical wavefunction renormalization for χ_- which also affects the coefficient in front of $|G_Z|^2$.

If the hypermultiplet masses vanish, $m_i = 0$, then the story is a little different. We may now set $\sigma = 0$ in the vacuum (recall that we assumed $\hat{\mathcal{W}}$ does not contain a linear piece, so $\sigma = 0$ is guaranteed to be a critical point). The full $\mathbf{CP}^{N_c-1}$ bosonic moduli space is now restored. This is in agreement with expectations from four dimensions, where we may happily construct any vortex string, built around the vacuum with $A = \tilde{Q}_i \equiv 0$. We conclude that the superpotential $\hat{\mathcal{W}}(A)$ does not affect the bosonic zero modes in this case, a point made previously in [12]. However, the superpotential does still affect the fermi zero modes. We now turn to a study of these.

4.4 Fermions

We will study the fermions in the case with vanishing hypermultiplet masses $m_i = 0$. Of all the Dirac equations in (2.25), only that for η is modified by the superpotential. It now reads

$$-\frac{i}{e^2}\,\bar{\not{D}}\eta - \frac{i\sqrt{2}}{e^2}[A,\bar{\lambda}] - \sqrt{2}\tilde{Q}_i^\dagger\bar{\psi}_i - \sqrt{2}\bar{\tilde{\psi}}_i Q_i^\dagger - \frac{\partial^2\hat{\mathcal{W}}(A)}{\partial A^2}\bar{\eta} = 0. \tag{4.16}$$

In the background of the vortex, we may again set $A = \tilde{Q}_i = 0$. This means that all right-moving fermi zero modes — those donated by λ_+ and $\bar{\psi}_{+i}$ — remain the same as in the $\mathcal{N} = (2,2)$ case, given by solutions to

$$\begin{aligned}\sqrt{2}\mathcal{D}_z\lambda_+ &= -e^2\,Q_i\bar{\psi}_{+i}\,,\\ \sqrt{2}\mathcal{D}_{\bar{z}}\bar{\psi}_{+i} &= -Q_i^\dagger\lambda_+.\end{aligned} \tag{4.17}$$

If the lowest order term in the superpotential $\hat{\mathcal{W}}$ is cubic or higher, then the left-moving fermi zero modes are similarly unaffected. However, if the superpotential $\hat{\mathcal{W}}(A)$ includes a quadratic mass term

$$\hat{\mathcal{W}}(A) = \mu_2 A^2 + \ldots \tag{4.18}$$

then the equations for the left moving fermi zero modes become

$$\begin{aligned}\sqrt{2}i\mathcal{D}_{\bar{z}}\eta_- &= -e^2\bar{\tilde{\psi}}_{-i}Q_i^\dagger - \sqrt{2}\mu_2\bar{\eta}_-\\ \sqrt{2}i\mathcal{D}_z\bar{\tilde{\psi}}_{-i} &= \eta_- Q_i.\end{aligned} \tag{4.19}$$

These equations are no longer related to the bosonic zero mode equations (2.16): this is to be expected since, in breaking to $\mathcal{N}=1$ supersymmetry, we have lost the half of supersymmetry which ensured the correspondence between bosonic zero modes and left-moving fermionic zero modes. Nevertheless, as stressed in [12], the Dirac equations (4.19) must still admit the same number of solutions as the equations with $\mu_2=0$. This follows from the fact that the zero modes are chiral on the worldsheet, and cannot gain a mass through a deformation. For a single $k=1$ vortex in the U(2) gauge theory, (4.19) was analyzed in [12], both perturbatively in $\mu_2\rho$, as well as in the large μ_2 limit.

To summarize, we learn that the deformation leaves the fermi zero modes untouched unless $\mu_2\neq 0$, in which case it deforms the profile of the left-moving fermi zero modes only. However, the number of zero modes on the worldsheet remains the same. Let us now compare this with the predictions from the proposed worldsheet deformation (4.11).

Implications for worldsheet fermions. In the presence of the superpotential $\hat{\mathcal{W}}$, the fermionic terms in the U(k) worldsheet theory read[11]

$$\begin{aligned}\mathcal{L}_{\rm fermi} = 2iT\ \mathrm{Tr}_k\left(\bar\chi_-\mathcal{D}_+\chi_- + \bar\chi_+\mathcal{D}_-\chi_+\right) + 2i\sum_{i=1}^{N_c}\left(\bar\xi_{-i}\mathcal{D}_+\xi_{-i}+\bar\xi_{+i}\mathcal{D}_-\xi_{+i}\right)\\ -\sqrt{2}\,\mathrm{Tr}_k\left([\bar\chi_-,[\sigma,\chi_+]]-[\bar\chi_+,[\bar\zeta_-,z]]-[\bar\xi_-,[\bar\zeta_+,z]]\right)+\text{h.c.}\\ -\sqrt{2}\sum_{i=1}^{N_c}\left(\bar\xi_{-i}\sigma\xi_{+i}-\bar\xi_{+i}\bar\zeta_-\phi_i+\bar\xi_{-i}\bar\zeta_+\phi_i\right)+\mathrm{Tr}_k\left(\chi_-\frac{\partial^2\hat{\mathcal{W}}(\sigma)}{\partial\sigma^2}\bar\zeta_+\right)+\text{h.c.}\ .\end{aligned} \tag{4.20}$$

The $\mathcal{N}=(0,2)$ superpotential is responsible for only the final term. Integrating out the auxiliary fermions $\zeta_\pm$ again gives constraints on the dynamical fermions,

$$\sum_i\phi_i\bar\xi_{+i}+[z,\bar\chi_+]=0 \quad\text{and}\quad \sum_i\phi_i\bar\xi_{-i}+[z,\bar\chi_-]=\frac{\partial^2\hat{\mathcal{W}}(\sigma)}{\partial\sigma^2}\chi_-. \tag{4.21}$$

We see that the right-moving fermions are unaffected by the superpotential, in agreement with the Dirac equations (4.17). Similarly, if $\hat{\mathcal{W}}$ has no quadratic term, so $\mu_2=0$, then the left-moving constraints are also left unchanged if we set $\sigma=0$ (we shall see the role played by a non-zero σ shortly). However, when $\mu_2\neq 0$, setting $\sigma=0$ still leaves deformed constraints on the left-moving fermions. For example, in the case of a single $k=1$ vortex, the constraints read

$$\sum_{i=1}^{N_c}\phi_i\bar\xi_{-i}=\mu_2\chi_-. \tag{4.22}$$

It's worth making a comment on this point. In the $\mathcal{N}=(0,2)$ theory, we have defined the left-moving worldsheet fermions such that their kinetic terms are diagonal: $\bar\chi_-\partial_+\chi_-+\bar\xi_{-i}\mathcal{D}_+\xi_{-i}$. The constraint (4.22) holds in this basis. It is always possible to redefine the

[11] As we mentioned previously, the deformation from $\mathcal{N}=2$ to $\mathcal{N}=1$ may also induce a finite wavefunction renormalization of the left-moving fermion kinetic terms. We will not consider this here.

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fermions so that the constraint (4.22) reverts to the original $\mathcal{N}=(2,2)$ constraint (2.38),

$$\bar{\xi}'_{-i} = \bar{\xi}_{-i} - \frac{\mu_2}{r}\bar{\phi}_i\chi_- \quad \Rightarrow \quad \sum_{i=1}^{N_c}\phi_i\,\bar{\xi}'_{-i} = 0. \tag{4.23}$$

This will then lead to a non-diagonal form for the fermion kinetic terms.

It was argued in [12] that, even in the presence of the four-dimensional superpotential $\hat{\mathcal{W}}(A) = \mu_2 A^2$, the worldsheet theory of the vortex string still retains $\mathcal{N}=(2,2)$ supersymmetry. This argument was based on the survival of the left-moving fermi zero modes, and the lack of a suitable $\mathcal{N}=(0,2)$ deformation of the $\mathbf{CP}^{N_c-1}$ sigma-model. We disagree with this conclusion. The vortex worldsheet theory is not described by a $\mathbf{CP}^{N_c-1}$ sigma-model, but rather by a $\mathbf{C}\times\mathbf{CP}^{N_c-1}$ sigma-model and, as we have seen, there is a suitable deformation of the latter in which χ_-, the left-moving fermion in $\mathbf{C}$, mixes with ξ_{-i}. Moreover, this mixing is necessary to correctly capture the bosonic properties of the vortex with arbitrary superpotential and masses. As we explained above, to see this mixing between χ_- and ξ_{-i} from an explicit analysis of the fermions would require us to solve the fermi zero mode equations (4.19), and take their overlap to determine both the kinetic terms and the constraint condition for the Grassmann collective coordinates of the vortex.

4.5 Symmetries and other aspects

We now discuss various further aspects of the worldsheet theory, starting with an analysis of the symmetries. We will show that the worldsheet superpotential has the correct properties under R-symmetry transformations to be induced by the superpotential $\hat{\mathcal{W}}(A)$. The addition of the superpotential $\hat{\mathcal{W}}(A)$ breaks both the $\mathrm{U}(1)_R$ and the $\mathrm{U}(1)_V$ symmetries in four dimensions. If the superpotential takes the form,

$$\hat{\mathcal{W}}(A) = \sum_{n=2}\mu_n A^n. \tag{4.24}$$

Treating the parameters μ_n as spurion fields, the symmetry is restored if μ_n carries charge $(2-2n,2)$ under $\mathrm{U}(1)_R\times\mathrm{U}(1)_V$. Let us check that these charges descend to the worldsheet theory. The deformation (4.11) once again destroys both $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_V$ on the worldsheet, this time through the presence of the worldsheet fermi interactions. The final term in (4.20) is,[12]

$$\sum_n n(n-1)\mu_n\ \mathrm{Tr}_k\left(\chi_-\sigma^{n-2}\,\bar{\zeta}_+\right). \tag{4.25}$$

Examining the table in section 2.4.1, we see that the $\mathrm{U}(1)_R\times\mathrm{U}(1)_V$ worldsheet symmetry is again restored if μ_n is assigned charges $(2-2n,2)$, in agreement with the analysis in four dimensions.

Note that the $\mathrm{U}(1)_Z$ symmetry on the worldsheet, which arises from rotational invariance in the $z=x^1+ix^2$ plane, is left unbroken by the deformation (4.25) as, indeed, it must be.

[12]The presence of $\bar{\zeta}_+$ in this expression, rather than ζ_+, is crucial in this analysis. It follows from the component expansion (3.22) and ultimately from the fact Σ arises from the decomposition of a $(2,2)$ twisted chiral multiplet as opposed to a $(2,2)$ chiral multiplet.

Discrete symmetries. One can also check that the deformation on the worldsheet is consistent with the discrete symmetries of the bulk theory.[13] We start by considering the action of parity, defined by

$$P : x^i \to -x^i \quad i = 1, 2, 3 \tag{4.26}$$

The original $\mathcal{N} = 2$ theory can be written in terms of Dirac spinors. For example, the adjoint Dirac spinor is $\Psi = (\lambda, \bar{\eta})^T$. Parity maps $P : \Psi \to \gamma^0 \Psi$, or

$$P : \lambda \leftrightarrow \bar{\eta} \quad \text{and} \quad P : \psi_i \leftrightarrow \bar{\tilde{\psi}}_i \tag{4.27}$$

while for the complex adjoint scalar $P : A \to A^\star$. (The imaginary part is really a pseudoscalar). Because the $z = x^1 + ix^2 \to -z$ part of the parity transformation can be undone by the rotation $\mathrm{U}(1)_Z$ on the worldsheet, we may restrict attention to the simpler parity transformation $P : x^3 \to -x^3$, with x^1 and x^2 untouched. This is the parity action under which the vortex string remains invariant. It must therefore descend to the worldsheet. Indeed, as we reviewed in section 2, (λ, ψ) donate right-moving zero modes χ_+ and ξ_{+i}, while $(\bar{\eta}, \bar{\tilde{\psi}})$ donate left-moving zero modes χ_- and ξ_{-i}. So the action of parity (4.27) in the bulk also exchanges left and right-movers on the worldsheet.

So much for the $\mathcal{N} = 2$ theory. What happens in the presence of the $\mathcal{N} = 1$ deformation? This pure parity symmetry (4.27) is broken in the 4d theory because the interactions of λ and η are different. This is also seen in our $\mathcal{N} = (0, 2)$ worldsheet theory where the interactions of left and right movers differ.

The 4d $\mathcal{N} = 2$ theory is also invariant under CP. Under charge conjugation, $C : B \to -B$ and the vortex is mapped onto the anti-vortex. So this cannot be a symmetry of the worldsheet. However, under the particular parity transformation

$$P' : x^2 \to -x^2 \tag{4.28}$$

with x^1 and x^3 invariant, we also have $B_3 \to -B_3$. Moreover, the complex coordinate z transverse to the vortex string is mapped to $P' : z \to z^\star$. This ensures that the bosonic vortex solution is invariant under CP'. For example, we have

$$\mathcal{D}_z Q_i \xrightarrow{C} \mathcal{D}_z Q^\dagger \xrightarrow{P'} \mathcal{D}_{\bar{z}} Q^\dagger \tag{4.29}$$

so the Bogomolnyi equation $\mathcal{D}_z Q = 0$ remains invariant under CP'. When acting on the fermions, CP' sends Weyl spinors to their complex conjugates,

$$CP' : \psi_i \to -i\sigma_2 \bar{\psi}_i \quad , \quad CP' : \lambda \to -i\sigma_2 \bar{\lambda} \quad , \quad \text{etc.} \tag{4.30}$$

This symmetry also descends to the worldsheet, where it acts as complex conjugation, as can be checked explicitly from the zero mode expressions of section 2. We have,

$$CP' : \phi_i \to \bar{\phi}_i \quad \text{and} \quad CP' : \xi_{\pm i} \to \bar{\xi}_{\pm i} \quad \text{etc} \tag{4.31}$$

[13] We thank M. Shifman and A. Yung for stressing the importance of this.

Note that, just as CP' in the 4d theory didn't exchange λ and $\bar{\eta}$, so this symmetry on the worldsheet doesn't send left-movers to right-movers. This can be traced to the fact that the action CP' under which the string is invariant doesn't affect x^3.

Unlike the pure parity transformation, the CP' symmetry survives the deformation to $\mathcal{N} = 1$ supersymmetry. More precisely, the symmetry survives if the parameters in the superpotential $\mathcal{W} = \mu_n A^n$ are real. Alternatively we can think of these parameters as transforming under $CP' : \mu_n \to \mu_n^\star$. The same behavior is seen in the worldsheet theory. Invariance of the final term in (4.20) requires that $CP' : \mu_n \to \mu_n^\star$, in agreement with the 4d analysis.

The four-fermi term. So far we have neglected the role of σ on the string worldsheet. In the $\mathcal{N} = (2,2)$ case, we saw that σ correctly takes into account the effect of the Yukawa couplings in four-dimensions, resulting in a four-fermi term (2.49) on the worldsheet. It will play the same role here. The equation of motion (2.50) for the adjoint field A is now changed by the superpotential $\hat{\mathcal{W}}(A)$. Even if the superpotential has $\mu_2 = 0$, so the profiles of both left and right-moving fermionic zero modes are the same as in the $\mathcal{N} = 2$ theory, the solutions to the full equations of motion, including Yukawa sources for A, will necessarily differ. We would expect this to feed back into the worldsheet dynamics. As in the $\mathcal{N} = (2,2)$ case, it is difficult to determine this explicitly, but thankfully the lifting of the zero modes is once again dictated by the symmetries of the problem.

Let's start by examining the simplest case, with $\hat{\mathcal{W}}(A) = \mu_2 A^2$, so that the fermionic constraint equation is given by (4.22). Integrating out σ on the worldsheet once again gives rise to a four-fermi term

$$\mathcal{L}_{4-\mathrm{fermi}} = -\frac{|\bar{\xi}_{-i}\xi_{+i}|^2}{(r + 2|\mu_2|^2/T)} \tag{4.32}$$

which, up to an overall rescaling, looks the same as the $\mathcal{N} = (2,2)$ four-fermi term (2.49). However this is deceptive, for the constraints (4.22) ensure that (4.32) now includes a component of χ_-. Previously, as we discussed in section 2.4.1, χ_- was prohibited from appearing in the four-fermi term since it was a Goldstino mode in the $\mathcal{N} = (2,2)$ theory. It loses this protection in the $\mathcal{N} = (0,2)$ theory.

If the superpotential contains quadratic and higher order terms, then integrating out σ results not only in a four-fermi term on the worldsheet, but also in a slew of higher order fermion lifting terms. These terms are an interesting prediction of the deformation (4.11).

A comment on anomalies. In section (2.5.2), we saw that additional fundamental $\mathcal{N} = 2$ hypermultiplets in four dimensions contributed extra zero modes to the vortex string which were captured in the gauged linear sigma model by adding $(N_f - N_c)$ chiral multiplets in the anti-fundamental representation of the $\mathrm{U}(k)$ worldsheet gauge group.

There exists a trivial generalization in the $\mathcal{N} = 1$ theories in which we add only four-dimensional chiral multiplets, instead of full hypermultiplets. For example, the addition of a single four dimensional chiral multiplet Q, transforming in the $\mathbf{N}_c$ of $\mathrm{U}(N_c)$, will contribute both bosonic and fermionic zero modes to the vortex string. These live in an $\mathcal{N} = (0,2)$ chiral multiplet $\tilde{\Phi}$ of the worldsheet theory, transforming in the $\bar{\mathbf{k}}$ of $\mathrm{U}(k)$. In contrast, the addition of $\tilde{Q}$, transforming in the $\bar{\mathbf{N}}_c$ of $\mathrm{U}(N_c)$, will contribute only fermi zero modes, living in an $\mathcal{N}=(0,2)$ fermi multiplet $\tilde{\Gamma}$ which transforms in the $\bar{\mathbf{k}}$ of $\mathrm{U}(k)$.

While the above observation is trivial, there is an interesting corollary in the quantum theory. The four-dimensional theory with unequal numbers of fundamental and anti-fundamental chiral multiplets is inconsistent at the quantum level, suffering a gauge anomaly. This inconsistency descends to the vortex worldsheet, which also suffers a $\mathrm{U}(k)$ gauge anomaly unless the number of chiral multiplets $\tilde{\Phi}$ is equal to the number of fermi multiplets $\tilde{\Gamma}$. It would be interesting to study vortices in chiral, anomaly free four-dimensional gauge theories, to see if there is a corresponding delicate anomaly cancellation on the vortex worldsheet.

The SQCD limit. To reach the $\mathcal{N}=1$ SQCD limit of the four-dimensional theory, we send $\mu_2 \to \infty$ to decouple the adjoint chiral multiplet A. On the worldsheet, this has the effect of decoupling the $\mathrm{U}(k)$ adjoint chiral multiplet Σ. At the same time, the constraint on the left-moving fermions (4.22) becomes simply $\chi_- = 0$, which effectively removes the fermi multiplet Ξ. The right-moving fermions on the worldsheet are still constrained to obey $\bar{\phi}\xi_{+i} = 0$ (in the case $N_f = N_c$) while the left-moving fermions ξ_{-i} are unconstrained. Nonetheless, the theory appears to be free of worldsheet gauge anomalies.

In this limit, the four-dimensional theory develops an enhanced, chiral flavor symmetry $S[\mathrm{U}(N_f)\times\mathrm{U}(N_f)]$, rotating left and right movers independently. (The "S" here is to remind us that the overall $\mathrm{U}(1)_B$ is part of the gauge group). In the presence of the FI parameter, this is broken spontaneously and the surviving symmetry in the vacuum is,

$$S[\mathrm{U}(N_c)\times\mathrm{U}(N_f-N_c)]\times\mathrm{U}(N_f)\times\mathrm{U}(1)_R \tag{4.33}$$

Here the $\mathrm{U}(1)_R$ is the anomaly-free R-symmetry. The same symmetry enhancement is also seen on the vortex worldsheet theory proposed above. There is once again a particular choice for the anomaly free R-current.

There is an issue with the normalizability of the fermi zero modes in this limit. As $\mu_2 \to 0$, the Dirac equation for the left-moving fermi zero modes become $\mathcal{D}_z\bar{\tilde{\psi}}_{-i} = 0$ which has only non-normalizable solutions. This could be mirrored on the worldsheet by infinite kinetic terms for Γ_i, of the type that we neglected in the discussion above. Alternatively, one could add a suitable deformation to the 4d theory, such as the meson field considered in [13], which once again renders these zero modes finite.

4.6 A D-brane construction

One can construct a D-brane configuration whose low-energy dynamics is governed by the four-dimensional theory of interest, namely $\mathcal{N}=2$ super QCD, broken to $\mathcal{N}=1$ by the addition of a superpotential $\hat{\mathcal{W}}(A)$ for the adjoint chiral multiplet. One starts with the usual Hanany-Witten set-up for four dimensional $\mathcal{N}=2$ gauge theories [26, 27]. This consists of two parallel NS5-branes lying in the 012345 directions and separated a distance $\Delta X^6 \sim l_s/e^2$ in the X^6 direction. The $\mathcal{N}=2$ $\mathrm{U}(N_c)$ gauge theory lives on N_c D4-branes, with worldvolume 01236, which are suspended between these two NS5-branes, while N_f D6-branes with worldvolume 0123789 provide the hypermultiplets. To describe the deformation (4.1) to $\mathcal{N} = 1$ supersymmetry, we introduce the complex coordinates

$$v = X^4 + iX^5 \quad , \quad w = X^8 + iX^9. \tag{4.34}$$

A superpotential $\hat{\mathcal{W}}(A)$ is induced on the D4-brane worldvolume if we bend the right-hand NS5-brane so that it no longer lies at the point $w = 0$, but rather on the complex curve [41]

$$w = \hat{\mathcal{W}}(v).$$

Note that in the limit $\mu_n \to \infty$, with μ_n defined in (4.24), the curved NS5-brane becomes multiple flat NS5-branes, lying a constant values of $v = X^4 + iX^5$, given by the roots of $\hat{\mathcal{W}}$. This is the description of the superpotential first presented in [42, 43].

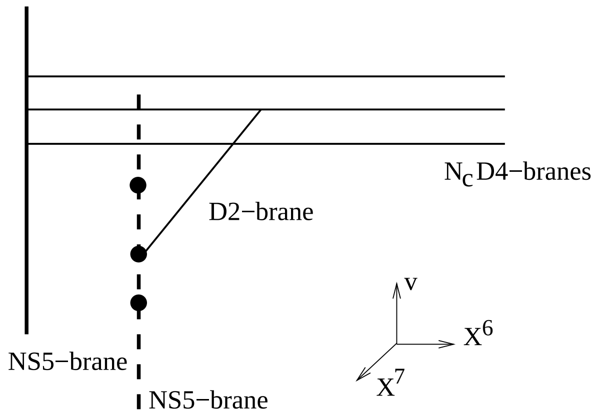

Figure 3: The deformed brane configuration.

We may now pass through the series of moves described in [1], turning on a FI parameter by separating the two NS5-branes in the X^7 direction, and identifying the vortices as stretched D2-branes. The final result is shown in figure 3 in the case of $N_f = N_c$. The figure shows a slice through $w = 0$. The dots depict the roots of $\hat{\mathcal{W}}(v)$, where the curved NS5-brane intersects the $w = 0$ plane; the ghostly dotted line shows where the NS5-brane has left this plane and is living at some other value of w. Figure 3 corresponds to a quartic superpotential, with three critical points. One can check that the theory on the D2-brane preserves $\mathcal{N} = (0, 2)$ supersymmetry. It is clear from the brane set-up that the D2-brane has a supersymmetric ground state only when it may safely stretch from the curved NS5-brane to a D4-brane, remaining at constant $v = m_i$ and without leaving the safety of $w = 0$. This requires

$$\left.\frac{\partial \hat{\mathcal{W}}(v)}{\partial v}\right|_{m_i} = 0. \tag{4.35}$$

This is the brane perspective on the statement that BPS vortices only exist when (4.35) is satisfied. It provides further evidence that a worldsheet superpotential of the form (4.11) is required.

4.7 A different superpotential

To end this section, we consider a different deformation of the $\mathcal{N} = 2$ theory which breaks the four dimensional supersymmetry to $\mathcal{N} = 1$. We add a superpotential of the form,

$$\mathcal{W}_{\mathcal{N}=2} = \sqrt{2} \sum_{i=1}^{N_f} \tilde{Q}_i \mathcal{V}_i(A) Q_i. \tag{4.36}$$

Here $\mathcal{V}_i(A)$ is an arbitrary holomorphic function of A. The four-dimensional quantum dynamics of theories of this type was previously studied in [44 − 46, 41]. We are here interested in the effect on the vortex worldsheet. In fact, we have already met one example of such a deformation that preserves $\mathcal{N} = 2$ supersymmetry, because the complex mass term is of this form with $\mathcal{V}_i(A) = A - m_i$. In that case, we saw that the effect was not to induce a superpotential on the worldsheet, but instead to change the relationship between $(0, 2)$ fermi and chiral fields,

$$\bar{\mathcal{D}}_+\Gamma_i = 2i\Sigma\Phi_i \;\longrightarrow\; \bar{\mathcal{D}}_+\Gamma_i = 2i(\Sigma - m_i)\Phi_i. \tag{4.37}$$

Given this, it is natural to conjecture that the general deformation (4.36) is captured by the worldsheet theory with the relationship,

$$\bar{\mathcal{D}}_+\Gamma_i = 2i\mathcal{V}_i(\Sigma)\Phi_i. \tag{4.38}$$

We will now provide evidence that this is indeed the case. We will show that the deformation (4.38) is in agreement with all symmetries of the theory, and reproduces the known behavior of the vortex. The details of the calculations are similar to those presented earlier, so we shall be brief.

Let us firstly study what becomes of the vortex. We take the vacuum of the four-dimensional theory to be

$$Q^a{}_i = v\delta^a{}_i \quad , \quad \tilde{Q}_i = 0 \qquad A = \mathrm{diag}(\nu_1, \ldots, \nu_{N_c}) \tag{4.39}$$

where ν_i is one of the roots of $\mathcal{V}_i$. If the ν_i are all distinct, the situation is the same as the one we encountered in section 2.5.1 with distinct masses m_i: there are N_c different vortices, each supported by the winding of a different Q_i. In contrast, if all ν_i coincide, the full $\mathbf{CP}^{N_c-1}$ internal moduli space of the vortex is restored.

Let us see how this is reproduced on the vortex worldsheet by the deformation (4.38). For definiteness, we take a single $k = 1$ vortex string in the $N_f = N_c$ theory. The bosonic part of the worldsheet theory is given by,

$$\mathcal{L}_{\mathrm{bose}} = T\,|\partial_m z|^2 + \sum_{a=1}^{N_c}\left(|\mathcal{D}_m\phi^a|^2 - 2|\mathcal{V}_i(\sigma)|^2|\phi_i|^2\right) + D\left(\sum_{i=1}^{N_c}|\phi_i|^2 - r\right) + \frac{\theta}{2\pi}u_{01}.$$

If the roots of ν_i of $\mathcal{V}_i(\sigma)$ are distinct, this theory has isolated vacua, given by

$$|\phi_j|^2 = r\delta_{ij} \quad , \quad \sigma = \nu_i. \tag{4.40}$$

However, there is an ambiguity here since $\mathcal{V}_i$ has multiple roots ν_i. Suppose, for definiteness, that $\mathcal{V}_i(\sigma)$ is a polynomial of degree P_i. Then it appears that, for each $i = 1, \ldots, N_c$, there are P_i different vacua of the worldsheet theory. How are we to interpret these? In past examples [18, 3, 4], different vacua of the worldsheet corresponded to different physical vortices — see section 2.5.1. But we certainly don't want the same interpretation here because the four-dimensional theory doesn't have P_i distinct vortices, each with Q_i winding asymptotically. Thankfully, the interpretation of the multiple worldsheet vacua in the present case is somewhat different. For fixed $i = 1, \ldots, N_c$, the P_i different vacua differ only in the value of the auxiliary field σ. The field σ is to be integrated out, set equal to its classical, algebraic equation of motion. But there are P_i different solutions to this algebraic equation. The theory is only complete if we specify which of these solutions we are to take. This means that the vacuum $\sigma = \nu_i$ chosen in (4.40) is not a dynamical variable, but rather a parameter of the worldsheet theory. We are therefore free to fix it as we please, and the only natural candidate is to equate it with the four-dimensional vacuum value ν_i in (4.39).[14] The end result is a situation where the same worldsheet Lagrangian describes the vortex string in different four-dimensional vacua; the specific four-dimensional vacuum of interest appears as a boundary condition on the auxiliary σ field.

As a check of the conjecture (4.38), we can confirm that the $\mathrm{U}(1)_R \times \mathrm{U}(1)_V$ charges are consistent. If we write the superpotential as

$$\mathcal{V}_i(A) = \sum_{n=0} h_n^{(i)} A^n \tag{4.41}$$

then we are required to assign spurion charge $(2-2n, 0)$ to $h_n^{(i)}$. Let's check that this is in agreement with the worldsheet. The deformation (4.38) gives rise to the terms

$$L_{\text{vortex}} = \ldots + \sqrt{2} \sum_n \, (n h_n^{(i)} \bar{\xi}_{-i} \sigma^{n-1} \phi_i \bar{\zeta}_+ \;+\; h_n^{(i)} \bar{\xi}_{-i} \sigma^n \xi_{+i}) + \ldots \tag{4.42}$$

from which we learn that $h_n^{(i)}$ must again be assigned charge $(2-2n, 0)$ under the worldsheet $\mathrm{U}(1)_R \times \mathrm{U}(1)_V$.

A. The alphabet

This appendix is included to help the reader keep track of the burgeoning conventions. The four dimensional fields are all components of $\mathcal{N} = 1$ superfields,

- A_μ : 4d gauge field in the vector multiplet V
- A : Adjoint valued 4d scalar in the chiral multiplet A
- Q_i : Fundamental 4d scalar in the chiral multiplet Q_i
- $\tilde{Q}_j$: Fundamental 4d scalar in the chiral multiplet $\tilde{Q}_j$
- λ : Adjoint valued 4d fermion in the vector multiplet V
- η : Adjoint valued 4d fermion in the chiral multiplet A
- ψ_i : Fundamental 4d fermion living in the chiral multiplet Q_i
- $\tilde{\psi}_j$: Anti-fundamental 4d fermion in the chiral multiplet $\tilde{Q}_j$.

[14] The equation of motion for σ includes a term bilinear in the fermions, seen explicitly in (4.42). The root of the equation of motion is taken to be the four-dimensional vacuum value ν_i when the fermions vanish, and is continuously connected to ν_i when the fermions turn on.

The worldsheet fields are all components of $\mathcal{N} = (0,2)$ superfields as described in section 3:

z : Worldsheet scalar arising from broken translational invariance, in the chiral multiplet Z
ϕ_i : Worldsheet scalar corresponding to orientation modes of the string, in the chiral multiplet Φ_i
σ : Worldsheet auxiliary scalar in the chiral multiplet Σ
u_m : Worldsheet gauge field in the vector multiplet U
χ_+ Worldsheet Goldstino fermion in the chiral multiplet Z
χ_- Worldsheet fermion in the fermion multiplet Ξ
ξ_{+i} : Worldsheet fermions living in the fermion multiplet Φ_i.
ξ_{-i} : Worldsheet fermions in the fermion multiplet Γ_i
$\bar{\zeta}_+$: Worldsheet auxiliary fermion living in the chiral multiplet Σ.
ζ_- : Worldsheet auxiliary fermion in the vector multiplet U.

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References

[1] A. Hanany and D. Tong, *Vortices, instantons and branes*, *JHEP* **07** (2003) 037 [hep-th/0306150].

[2] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, *Nonabelian superconductors: vortices and confinement in $N = 2$ SQCD*, *Nucl. Phys.* **B 673** (2003) 187 [hep-th/0307287].

[3] M. Shifman and A. Yung, *Non-abelian string junctions as confined monopoles*, *Phys. Rev.* **D 70** (2004) 045004 [hep-th/0403149].

[4] A. Hanany and D. Tong, *Vortex strings and four-dimensional gauge dynamics*, *JHEP* **04** (2004) 066 [hep-th/0403158].

[5] N. Dorey, *The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms*, *JHEP* **11** (1998) 005 [hep-th/9806056].

[6] N. Dorey, T.J. Hollowood and D. Tong, *The BPS spectra of gauge theories in two and four dimensions*, *JHEP* **05** (1999) 006 [hep-th/9902134].

[7] D. Tong, *Superconformal vortex strings*, *JHEP* **12** (2006) 051 [hep-th/0610214].

[8] D. Tong, *TASI lectures on solitons*, hep-th/0509216.

[9] V. Markov, A. Marshakov and A. Yung, *Non-abelian vortices in $N = 1^*$ gauge theory*, *Nucl. Phys.* **B 709** (2005) 267 [hep-th/0408235].

[10] A. Gorsky, M. Shifman and A. Yung, *Non-abelian Meissner effect in Yang-Mills theories at weak coupling*, *Phys. Rev.* **D 71** (2005) 045010 [hep-th/0412082].

[11] S. Bolognesi, *The holomorphic tension of vortices*, *JHEP* **01** (2005) 044 [hep-th/0411075]; *The holomorphic tension of nonabelian vortices and the quark = dual-quark condensate*, *Nucl. Phys.* **B 719** (2005) 67 [hep-th/0412241].

[12] M. Shifman and A. Yung, *Non-abelian flux tubes in SQCD: supersizing world-sheet supersymmetry*, *Phys. Rev.* **D 72** (2005) 085017 [hep-th/0501211].

[13] A. Gorsky, M. Shifman and A. Yung, *N = 1 supersymmetric quantum chromodynamics: how confined non-abelian monopoles emerge from quark condensation*, *Phys. Rev.* **D 75** (2007) 065032 [hep-th/0701040].

[14] H.B. Nielsen and P. Olesen, *Vortex-line models for dual strings*, *Nucl. Phys.* **B 61** (1973) 45.

[15] R. Jackiw and P. Rossi, *Zero modes of the vortex-fermion system*, *Nucl. Phys.* **B 190** (1981) 681.

[16] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton Univ. Pr. (1992).

[17] S. Vandoren and P. van Nieuwenhuizen, *New instantons in the double-well potential*, *Phys. Lett.* **B 499** (2001) 280 [hep-th/0010130].

[18] D. Tong, *Monopoles in the Higgs phase*, *Phys. Rev.* **D 69** (2004) 065003 [hep-th/0307302].

[19] A. Hanany and K. Hori, *Branes and $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 513** (1998) 119 [hep-th/9707192].

[20] R.S. Ward, *Slowly moving lumps in the CP^1 model in $(2+1)$-dimensions*, *Phys. Lett.* **B 158** (1985) 424.

[21] R.A. Leese and T.M. Samols, *Interaction of semilocal vortices*, *Nucl. Phys.* **B 396** (1993) 639.

[22] M. Shifman and A. Yung, *Non-abelian semilocal strings in $N = 2$ supersymmetric QCD*, *Phys. Rev.* **D 73** (2006) 125012 [hep-th/0603134].

[23] A. Achucarro and T. Vachaspati, *Semilocal and electroweak strings*, *Phys. Rept.* **327** (2000) 347 [*Phys. Rept.* **327** (2000) 427] [hep-ph/9904229].

[24] A.A. Penin, V.A. Rubakov, P.G. Tinyakov and S.V. Troitsky, *What becomes of vortices in theories with flat directions*, *Phys. Lett.* **B 389** (1996) 13 [hep-ph/9609257].

[25] A. Achucarro, A.C. Davis, M. Pickles and J. Urrestilla, *Vortices in theories with flat directions*, *Phys. Rev.* **D 66** (2002) 105013 [hep-th/0109097].

[26] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys.* **B 492** (1997) 152 [hep-th/9611230].

[27] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166].

[28] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Moduli space of non-abelian vortices*, *Phys. Rev. Lett.* **96** (2006) 161601 [hep-th/0511088].

[29] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Solitons in the Higgs phase: the moduli matrix approach*, *J. Phys.* **A39** (2006) R315 [hep-th/0602170].

[30] N.S. Manton, *A remark on the scattering of BPS monopoles*, *Phys. Lett.* **B 110** (1982) 54.

[31] T.M. Samols, *Vortex scattering*, *Commun. Math. Phys.* **145** (1992) 149.

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[32] R. Auzzi, M. Shifman and A. Yung, *Composite non-abelian flux tubes in $N = 2$ SQCD*, *Phys. Rev.* **D 73** (2006) 105012 [hep-th/0511150].

[33] K. Hashimoto and D. Tong, *Reconnection of non-abelian cosmic strings*, *JCAP* **09** (2005) 004 [hep-th/0506022].

[34] M. Eto et al., *Universal reconnection of non-abelian cosmic strings*, *Phys. Rev. Lett.* **98** (2007) 091602 [hep-th/0609214].

[35] C.M. Hull and E. Witten, *Supersymmetric σ-models and the heterotic string*, *Phys. Lett.* **B 160** (1985) 398.

[36] M. Dine and N. Seiberg, $(2,0)$ *superspace*, *Phys. Lett.* **B 180** (1986) 364.

[37] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[38] A. Adams, A. Basu and S. Sethi, $(0,2)$ *duality*, *Adv. Theor. Math. Phys.* **7** (2004) 865 [hep-th/0309226].

[39] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[40] S.C. Davis, A.-C. Davis and M. Trodden, *$N = 1$ supersymmetric cosmic strings*, *Phys. Lett.* **B 405** (1997) 257 [hep-ph/9702360].

[41] J. de Boer and Y. Oz, *Monopole condensation and confining phase of $N = 1$ gauge theories via M-theory fivebrane*, *Nucl. Phys.* **B 511** (1998) 155 [hep-th/9708044].

[42] S. Elitzur, A. Giveon and D. Kutasov, *Branes and $N = 1$ duality in string theory*, *Phys. Lett.* **B 400** (1997) 269 [hep-th/9702014].

[43] S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici and A. Schwimmer, *Brane dynamics and $N = 1$ supersymmetric gauge theory*, *Nucl. Phys.* **B 505** (1997) 202 [hep-th/9704104].

[44] A. Kapustin, *The coulomb branch of $N = 1$ supersymmetric gauge theory with adjoint and fundamental matter*, *Phys. Lett.* **B 398** (1997) 104 [hep-th/9611049].

[45] T. Kitao, S. Terashima and S.-K. Yang, *$N = 2$ curves and a coulomb phase in $N = 1$ SUSY gauge theories with adjoint and fundamental matters*, *Phys. Lett.* **B 399** (1997) 75 [hep-th/9701009].

[46] A. Giveon, O. Pelc and E. Rabinovici, *The Coulomb phase in $N = 1$ gauge theories with an LG-type superpotential*, *Nucl. Phys.* **B 499** (1997) 100 [hep-th/9701045].